MATH 442 — HOMEWORK 5

DUE: SEP. 28, 2005

Section 2.4: **# 19 (a)**. And then guess a solution formula for the diffusion equation

$$u_t = k\Delta u \tag{1}$$

in two dimensions, under the initial condition $u(x, y, 0) = \phi(x, y)$. Show your formula is valid (meaning it satisfies the diffusion equation (1) and the initial condition) in the special case where ϕ has separated form $\phi(x, y) = f(x)g(y)$.

Additional Problem A (Diffusion equation with lower order terms.) Suppose

$$u_t - ku_{xx} + au_x + bu = 0,$$

where $k > 0,\ a, b \in \mathbb{R}$ are constants. Take initial condition $u(x, 0) = \phi(x)$.

Let $v(x, t) = e^{bt}u(x + at, t)$ and find the PDE satisfied by v. Solve for v, then solve for u.

Remarks.

(1) This problem essentially combines Exercises 2.4.16 and 2.4.18.
(2) The "convection" term au_x in the diffusion equation indicates that the diffusion takes place in a moving medium (for example in water flowing down a straight river). Where does this moving coordinate frame show up in your solution formula for u?

Section 2.4: **# 17.**

Additional Problem B: (Wave equation with lower order terms.) Consider the wave equation

$$u_{tt} - c^2u_{xx} + \alpha u_t + \beta u_x + \gamma u = 0$$

for some constants $\alpha,\ \beta,\ \gamma$ (which correspond physically to dissipation/friction, transport, restoring force). Make the change of variable

$$v(x, t) = \exp\left[(\alpha t - \beta c^{-2}x)/2\right]u(x, t)$$

and deduce the PDE satisfied by v. (You need not solve this PDE.)

Additional Problem C: Suppose

$$v_{tt} - c^2v_{xx} + \delta^2 v = 0, \quad -\infty < x < \infty,$$

for some constant $\delta \in \mathbb{R}$. Write

$$w(x, y, t) = \cos(c^{-1}\delta y)v(x, t)$$

and deduce a PDE satisfied by w (it will involve $x,\ y$ and t derivatives).

Additional Problem D: Suppose

$$v_{tt} - c^2v_{xx} - \delta^2 v = 0, \quad -\infty < x < \infty.$$

Find a change of variable that reduces to a PDE in $x,\ y,\ t$ similar to Probelm C.

Conclusions. Additional Problems B, C and D show how the one dimensional wave equation with lower order terms can always be reduced to the two dimensional wave equation. (And Section 9.2 in the textbook show how to solve the two dimensional wave equation.)