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On the Lattice Diameter of a Convex Polygon [1]

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Abstract

The lattice diameter, $\ell(P)$, of a convex polygon P in R^2 measures the longest string of integer points on a line contained in P. We relate the lattice diameter to the area and to the lattice width of P, $w_L(P)$. We show, e.g., that $w_L \leq (4/3)\ell + 1$, thus giving a discrete analogue of Blaschke's theorem.

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1 The Area of Lattice Polygons

Let P be a convex, closed, non-empty lattice polygon, i.e., $P = \operatorname{conv}(P \cap \mathbb{Z}^2)$. The *lattice diameter*, $\ell(P)$, measures the longest string of integer points on a line contained in P

$$\ell(P) = \max\{|P \cap \mathbb{Z}^2 \cap L| - 1 \ :\ L \text{ is a line}\}\ .$$

Thus $\ell(P) = 0$ if and only if P consists of a single lattice point, and the for the square $Q^1 = [0, \ell] \times [0, \ell]$ and for the special pentagon $Q^2 = \operatorname{conv}(\{(x, y) \in \mathbb{Z}^2 \ :\ 0 \le x, y \le \ell\} \cup \{(\ell+1, \ell+1)\} \setminus \{(0, 0)\})$ (for $\ell \in \mathbb{Z}^+$) one has $\ell(Q^1) = \ell(Q^2) = \ell$. (See Figure 1.) This definition is due to Stolarsky and Corzatt [5] who proved several properties of $\ell(P)$. The lattice diameter is invariant under the group of unimodular affine transformations $SL(2, \mathbb{Z})$; these are lattice preserving mappings $R^2 \to R^2$ also preserving parallel lines and area.

Figure 1

A simple consequence of the definition is the following fact on lattice points contained in P which first appeared in the literature in Rabinowitz [10].

$$(P \cap \mathbb{Z}^2) \cap ((\ell(P) + 1)z + (P \cap \mathbb{Z}^2)) = \emptyset \ \text{ for every } \ z \in \mathbb{Z}^2,\ z \ne (0, 0)\ . \tag{1}$$

To see this we note that the common point to P, $\mathbb{Z}^2$, and $(\ell(P) + 1)z + P$ would be of the form $(\ell(P) + 1)z + x$ with $x \in (P \cap \mathbb{Z}^2)$ implying that the string of $\ell(P) + 2$ integer points x, $x + z$, ..., $x + (\ell(P) + 1)z$ all belong to P contradicting the definition of the lattice diameter. Equation (1) implies that $\{(\ell(P) + 1)z + (P \cap \mathbb{Z}^2)\}_{z \in \mathbb{Z}^2}$ form a “packing” in $\mathbb{Z}^2$ which shows, in turn, that P contains at most $(\ell(P) + 1)^2$ lattice points,

$$|P \cap \mathbb{Z}^2| \le (\ell(P) + 1)^2\ . \tag{2}$$

An elementary argument and (1) imply that $(\ell(P) + 1)\mathbb{Z}^2 + P$ is a packing in R^2 by translates of P so that

$$\text{Area}\,(P) \le (\ell(P) + 1)^2\ . \tag{3}$$

For higher dimension the volume of P is not bounded by a function of $\ell(P)$; there are empty simplices $S \subset R^d$ (i.e., $S \cap \mathbb{Z}^d = \mathrm{vert}\,(S)$) having arbitrarily large volume (see Reeve [11], Bell [3], Scarf [12]), e.g., one can take (in R^3) $S = \mathrm{conv}\;(\{(0,0,0),(1,0,0),(0,1,0),(1,1,k)\})$.

Let $\mathrm{A}(k)$ denote the maximal area a convex lattice polygon P with $\ell(P) \le k$ can have. The square, i.e., Example Q^1, implies $\mathrm{A}(k) \ge k^2$. Alarcon [1] observed that this is far from being optimal, $\mathrm{Area}\,(Q^2) = k^2 + k - 1/2$. He also showed $\mathrm{A}(1) = 1.5$, $\mathrm{A}(2) = 5.5$, $\mathrm{A}(3) = 11.5$ and $\mathrm{A}(4) = 21$, and improved (3) to $\mathrm{A}(k) \le k^2 + 2k - 2$ for $k \ge 5$. Our first result is that $\mathrm{A}(k)$ is very close to the upper bound (3).

Theorem 1: *For $k \ge 5$ there exists a convex lattice polygon Q^3 with $\ell(Q^3) = k$ and* $\mathrm{Area}\,(Q^3) = k^2 + 2k - 4$.

Figure 2

The construction $Q^3 = Q^3(k)$ is an octagon with vertices $(-1,0)$, $(0,k-1)$, $(2,k)$, $(k-1,k+1)$, $(k+1,k)$, $(k,1)$, $(k-2,0)$, and $(1,-1)$, see Figure 2. In fact, for $k > 5$ the polygon Q^3 is indeed an octagon with only these eight vertices on its boundary and with $(k+1)^2 - 8$ interior points. For $k = 5$ two of its boundary points, $(2,k)$ and $(k-2,0)$, are not vertices, it becomes a hexagon. Thus Pick's theorem on the area of lattice polygons, i.e.,

$$\mathrm{Area}\,(P) = |\mathrm{int}(P) \cap \mathbb{Z}^2| - 1 + \frac{|\partial(P) \cap \mathbb{Z}^2|}{2}$$

implies $\mathrm{Area}\,(Q^3) = (k^2 + 2k - 7) - 1 + \frac{8}{2}$, as claimed. (This can be shown directly as well.) Alarcon's improvement of (3) also utilizes Pick's theorem, he shows that a maximal P has at least 4 vertices. We CONJECTURE that Q^3 is extremal, $\mathrm{A}(k) = k^2 + 2k - 4$.

2 Slopes of Diameters

Bang [2] solved Tarski's plank problem by showing that if a compact convex set in R^2 can be covered by n strips of widths $w_1, w_2, \dots, w_n$ then it can be covered with one strip of width $\sum_{1 \le i \le n} w_i$. Corzatt [5] CONJECTURED the following discrete analogue. If the set of lattice points contained in the lattice polygon P can be covered by n lines, $(P \cap \mathbb{Z}^2) \subset (L_1 \cup L_2 \cup \ldots \cup L_n)$, then there exists a set of covering lines $\mathcal{L} = \{L'_1, \dots, L'_n\}$, $(P \cap \mathbb{Z}^2) \subset (L'_1 \cup L'_2 \cup \ldots \cup L'_n)$ such that the lines in $\mathcal{L}'$ have at most four different slopes. This problem motivated Alarcon [1] to ask the maximum number of diameter directions of a lattice polygon.

A non-zero vector $u \in \mathbb{Z}^2$ is a *diameter direction* for the convex lattice polygon P if there is an integer z such that z, $z + u$, ..., $z + \ell(P)u$ all belong to P. Such a u is necessarily a *primitive* vector, i.e., its coordinates are coprime. Write $N(P)$ for the number of diameter directions of P. The triangle with vertices $(-1, -1)$, $(1, 0)$, $(0, 1)$ and baricenter $(0, 0)$ has 6 different diameter directions. Here we prove that

$$N(P) \le 4$$

for all convex lattice polygons with $\ell(P) > 1$. This is done by a good description (Theorem 2 below) of convex lattice polygons P that are maximal to containment with respect to $\ell(P) = \ell$.

Write $\mathcal{M}_\ell$ for the collection of maximal convex lattice polygons, i.e., $P \in \mathcal{M}_\ell$ if $\ell(P) = \ell$, and for any convex lattice polygon P' properly containing P, $\ell(P') > \ell$. One more definition: given primitive vectors u, $b \in \mathbb{Z}^2$ (nonparallel) and $z \in \mathbb{Z}^2$, the *half-open slab* $S(u, b, z)$ is defined as

$$S(u, b, z) = \{z + \alpha u + \beta b \ : \ 0 \le \alpha < \ell + 1, \ -\infty < \beta < +\infty\} \ .$$

THEOREM 2: *If $P \in \mathcal{M}_\ell$ then one of the following 3 cases holds.*

(i) *P has exactly two diameter directions, u_1 and u_2, say. They form a basis of $\mathbb{Z}^2$. Further, there are points z_1, $z_2 \in \mathbb{Z}^2$ and primitive vectors b_1 and b_2 such that z_i, $z_i + u_i$, ..., $z_i + \ell u_i \in P$ and*

$$P = \mathrm{conv}\Big(\mathbb{Z}^2 \cap S(u_1, b_1, z_1) \cap S(u_2, b_2, z_2)\Big). \tag{4}$$

(ii) *P has exactly three diameter directions, u_1, u_2, u_3. Any two of them form a basis of $\mathbb{Z}^2$ thus $u_3 = \pm u_1 \pm u_2$. Further, there are points $z_i \in \mathbb{Z}^2$ and primitive vectors b_i*

$(i = 1, 2, 3)$ *such that* $z_i, z_i + u_i, \ldots, z_i + \ell u_i \in P$ *and*

$$P = \operatorname{conv}\Big(\mathbb{Z}^2 \cap \bigcap_{1\le i\le 3} S(u_i, b_i, z_i)\Big) \tag{5}$$

(iii) *P has exactly four diameter directions. Then* (mod $SL(2,\mathbb{Z})$)*, i.e., up to a lattice preserving affine transformation) P is either the square Q^1 or the special pentagon Q^2. (See again Figure 1.)*

The proof is postponed to Section 4.

3 Width and Covering Radius

The lattice diameter is the natural counterpart of the *lattice width*, $w_L(P)$, which is defined as

$$w_L(P) = \min_{u\in\mathbb{Z}^2\ u\neq(0,0)} \left(\max_{x,y\in P} u(x-y)\right).$$

The lattice width is also invariant under the group of unimodular affine transformations $SL(2,\mathbb{Z})$. Thus $w_L(P) = 0$ if and only if P can be covered by a single line. For the square we have $w_L(Q^1) = \ell$ and for the special pentagon Q^2 in Example 1 we have $w_L(Q^2) = \ell + 1 > \ell(Q^2) = \ell$. In general, in Section 5 we prove the following consequence of Theorem 2.

THEOREM 3: $w_L(P) \le \lfloor \frac{4}{3}\ell(P)\rfloor + 1$ *and for given ℓ this upper bound is best possible.*

The following example, Q^4, shows that here equality can hold if ℓ is of the form $3t+1$. The polygon $Q^4 = Q^4(t)$ is a triangle with vertices $(0,0)$, $(4t+2, 2t+1)$, and $(2t+1, 4t+2)$; it has lattice diameter $\ell = 3t+1$ and lattice width $w_L(Q^4) = 4t+2$. For the other values of ℓ we obtain equality by considering the triangle $(0,0)$, $(t, 2t+1)$, $(2t+1, t+1)$. Its width is $2t+1$ and its diameter is $\lfloor (3t+1)/2\rfloor$.

The following example, Q^5, shows that there are other completely different polygons with almost equality in Theorem 3. Let $Q^5 = Q^5(\ell)$ be a hexagon with vertices $(0,0)$, $\left(\frac{1}{3}\ell, -\frac{1}{3}\ell\right)$, $(\ell, 0)$, $\left(\frac{4}{3}\ell, \frac{2}{3}\ell\right)$, (ℓ,ℓ), and $\left(\frac{1}{3}\ell, \frac{2}{3}\ell\right)$. We have $\ell(Q^5) = \ell$, and $w_L(Q^5) = \frac{4}{3}\ell$ for every $\ell \in \mathbb{Z}^+$, ℓ is divisible by 3.

Uwe Schnell [13] showed (in a slightly different form) another upper bound for the lattice width of an arbitrary convex, closed planar region C

$$w_L(C) \le \frac{4}{3}\text{Area}\,(C)\,\mu_2(C)\,, \tag{6}$$

where $\mu_2 := \mu_2(C)$ is the *covering radius*, i.e., the smallest positive real x such that the union of the regions of the form $z + xC$ for $z \in \mathbb{Z}^2$ covers the plane. For more about covering minima see Kannan and Lovász [8], or the survey of Gritzmann and Wills [7].

Although (6) frequently gives a better bound than Theorem 3, there are several examples, like Q^6 below, when $\ell(P)$ is smaller than Area $(P)\mu_2(P)$. Let $Q^6 = Q^6(t)$ be a tilted square of side length $\sqrt{160}\,t$ with vertices $(t, -3t)$, $(13t, t)$, $(9t, 13t)$, $(-3t, 9t)$, where $t \in \mathbb{Z}^+$. It contains the inscribed square $(0, 0)$, $(10t, 0)$, $(10t, 10t)$, $(0, 10t)$ and its covering radius is $\mu_2 = 1/(10t)$. On the other hand, it is easy to see that Area $(Q^6)\mu_2 = 16t$ is at least 1.2 times larger than $\ell(Q^6) = \lfloor (40/3)t \rfloor$. We CONJECTURE that in general Schnell's bound is at most $(1 + \sqrt{2})/2 = 1.207\ldots$ times larger than $\ell(C)$.

Another upper bound for the lattice width is due to L. Fejes-Tóth and Makai Jr. [6]

$$w_L(C) \leq \sqrt{\frac{8}{3}\text{Area}\,(C)}\,. \tag{7}$$

This is also sharp for some cases, like for the triangle $(0, t), (t, 0), (-t, -t)$, but again Q^6 shows that it could exceed the bound of Theorem 3 by more than 50 %.

4 The Maximal Polygons, the Proof of Theorem 2

We start with a statement that applies to every convex lattice polygon.

LEMMA 1: *Assume P is a convex lattice polygon and $u \in \mathbb{Z}^2$, $u \neq (0, 0)$. Then there is a longest segment $[z, v]$ contained in P and parallel with u such that z is a vertex of P. Further, for every such longest segment $[z, v]$, v lies on an edge $[v_1 v_2]$ of P so that the line through z and parallel with $[v_1 v_2]$ is tangent to P.*

The *proof* is simple and can be found in [4]. □

Consider now $P \in \mathcal{M}_\ell$ (with $\ell \geq 1$) and let u be a diameter direction for P. Apply Lemma 1 to get a longest segment $[z, v]$ with z a vertex. As $[z, v]$ is a longest segment in direction u, z, $z + u$, ..., $z + \ell u \in P \cap \mathbb{Z}^2$. Thus $[z, v]$ contains a lattice diameter.

Applying a suitable lattice preserving affine transformation we may assume $u = (0, 1)$, $z = (0, 0)$ and $v_2 - v_1 = b = (b_x, b_y)$ with $0 \leq 2b_y \leq b_x$, here $[v_1, v_2]$ is the edge of P specified by Lemma 1. We conclude that P lies in the half-open slab $S(u, b, z)$, see Figure 3.

As the area of the z, v_1, v_2 triangle is at most Area $(P) \leq (\ell + 1)^2$ by (3) and the area of the $z + (\ell + 1)u, v_1, v_2$ triangle is at least 1/2, we obtain that P is contained in the slightly narrower half open slab

$$S'(u,b,z) := \{z + \alpha u + \beta b \ : \ 0 \le \alpha < \ell + 1 - \frac{1}{2\ell+2}, \ -\infty < \beta < +\infty\} \ . \tag{8}$$

Figure 3

It follows from (1) that $(\pm(\ell+1), k) \notin P$ for all $k \in \mathbb{Z}$. Assume now that some $q = (q_x, q_y) \in \mathbb{Z}^2$ with $q_x > \ell + 1$ belongs to P. The triangle $T := \mathrm{conv}\{(0,0), (0,\ell), q\}$ meets the line $x = \ell + 1$ in a segment of length $\ell(q_x - \ell - 1)/q_x$. This segment must be lattice point free, so its length is less than 1, implying $q_x < \ell + 3$ for $\ell > 2$. The case $\ell \le 2$ is obvious, so from now on we always suppose $\ell > 2$. A simple computation reveals that the T contains a lattice point from the line $x = \ell + 1$ unless $q = (\ell + 2, \ell + 1)$.

We treat first this case $q = (\ell + 2, \ell + 1) \in P$ (which leads to case (iii) as we shall see soon). First $\mathrm{conv}\{(0,0), v, q\} \subset P$ shows $(0,\ell)$, $(1,\ell)$, $\ldots$, $(\ell,\ell) \in P$ and $(0,0)$, $(1,1)$, ..., $(\ell,\ell) \in P$. So (0,1), (1,0) and (1,1) are diameter directions. As the line $x = \ell + 1$ contains no lattice point of P we have $(\ell+1, \ell+1)$ and $(\ell+1, \ell) \notin P \cap \mathbb{Z}^2$. As $(\ell+2, \ell+1) \in P$ this implies that $(k, \ell+1) \notin P$ and $(k, k-1) \notin P$ for all $k \le \ell + 1$. We obtain that for all $(x,y) \in P \cap \mathbb{Z}^2$ other than $(\ell+2, \ell+1)$ we have $y \le \ell$ and $x \le y$. Further, $(k, -1) \notin P$ and $(k, \ell+1+k) \notin P$ for all $k \in \{-1, -2, ..., -(\ell+1)\}$. Also, $(-\ell, 0) \notin P$ since otherwise $(-\ell, 0)$, $(-\ell+2, 1)$, $\ldots$, $(\ell+2, \ell+1)$ all belong to P implying $\ell(P) > \ell$. Figure 4 shows the room left for P after these restrictions.

Figure 4

The maximality of P implies now that P equals $\operatorname{conv}\{(\ell+2,\ell+1),\ (0,0),\ (-\ell+1,0),\ (-\ell+1,1),\ (0,\ell)\}$. This is one of the special cases of (iii), the lattice preserving affine transformation $(x,y)\to(x-y+\ell,y)$ carries P to the "almost-square" special pentagon Q^2 of Figure 1.

From now on we assume that $|x|\le\ell$ for all $(x,y)\in P$. Thus P is confined to the parallelogram of Figure 3 bounded by the lines $x=\pm\ell$ and two other lines parallel to b. There are only six lattice directions in this parallelogram which can have a chord containing $\ell+1$ integer points. They are $(0,1)$, $(1,0)$, $(1,1)$, $(1,-1)$, $(2,1)$ and $(2,-1)$. To simplify matters we state

Claim 1: If u_1 and u_2 are diameter directions with $\det(u_1,\,u_2)=2$ of $P\in\mathcal{M}_\ell$ then the diameter segments $[z_1,\ z_1+\ell u_1]$ and $[z_2,\ z_2+\ell u_2]$ meet either at their midpoints or one segment is off by u_i. In these cases (mod $SL(2,\mathbb{Z}^2)$) P is either the square Q^1 or the almost square, Q^2, cf. Figure 1.

Proof. As we have seen above, we may suppose that $u_1=(0,1)$, $P\subset Q$ as in Figure 3 and $u_2=(2,1)$ or $u_2=(2,-1)$. The latter case leads to the square with vertices $(-\ell,\ell)$, $(0,0)$, $(\ell,0)$, and $(0,\ell)$. When $u_2=(2,1)$ the diameters are $\{(0,0),(0,1),\ldots,(0,\ell)\}$ and $\{(-\ell,i),(-\ell+2,i+1),\ldots,(\ell,i+\ell)\}\subset P$. Considering the string of $\ell+2$ lattice points from $(-1,i-1)$ to $(\ell,i+\ell)$ it follows that $(-1,i-1)\notin P$. Since $(-1,1)\in\operatorname{conv}((-\ell,i),(0,0),(0,\ell))\subset$

P, it follows $i \leq 1$. Using a symmetric argument we obtain that $i \in \{-1, 0, 1\}$ and can finish the proof as in the case $q = (\ell+2, \ell+1) \in P$ above. $\square$

Assume now that P has exactly k diameter directions, $u_1, \ldots, u_k$. Assume that P is not affinely equivalent to Q^1 neither Q^2. Then by the above Claim $\det(u_i, u_j) = \pm 1$ for any two diameter directions. This implies that $k \leq 3$. The diameters are z_i, $z_i + u_i$, ..., $z_i + \ell u_i$ ($i = 1, \ldots, k$) with suitable directions b_i of the edge opposite to z_i of P (see Lemma 1). Define

$$Q = \bigcap_{1 \leq i \leq k} S(u_i, b_i, z_i)\,.$$

Clearly $P \subset Q$. We claim $\ell(Q) = \ell$, so again by the maximality of P, $P = \mathrm{conv}(Q \cap \mathbb{Z}^2)$, finishing the proof.

Assume, on the contrary, that there exists a lattice point $q \in (Q \setminus P)$, and suppose that among these points q is one of the closest to P. Add this point to P, consider $P' := \mathrm{conv}\,(P \cup \{q\})$. So q is the only new lattice point in P', $P' \cap \mathbb{Z}^2 = P \cap \mathbb{Z}^2 \cup \{q\}$. The maximality of P implies that $\ell(P') > \ell(P)$, thus q creates a new longer diameter segment q, $q + u$, ..., $q + (\ell+1)u \in P' \cap \mathbb{Z}^2$ with $u \neq (0,0)$. As $\ell + 1$ of these points belong to P, we obtain that u is a diameter direction of P, too. However $S(u_i, b_i, z_i)$ contains no segments of direction u_i longer than ℓ. Thus u has to be different from $u_1, \ldots, u_k$, contradicting that P has exactly k diameter directions. Evidently, since P is not infinite, there are at least two diameter directions, $k = 2$ or 3. $\square$

5 Bounding the Width, the Proof of Theorem 3

As w_L is an integer for a lattice polygon we have to prove only $w_L < (4/3)(\ell + 1)$.

We give a sketch for the convex set

$$Q = \bigcap_{1 \leq i \leq k} S_i$$

where $S_i = S'(u_i, b_i, z_i)$ are the half-open slabs in (4) and (5) of Theorem 2 modified in (8). Denote the lattice width of the slabs by L. By (8) we have $L = \ell + 1 - 1/(2\ell + 2) < \ell + 1$.

Applying a suitable $SL(2, \mathbb{Z}^2)$ mapping we may assume that $u_1 = (1, 0)$, $u_2 = (0, 1)$ and u_3, if exists, is $(1, 1)$ or $(-1, -1)$. We will use the fact (which is easy to establish) that the lattice width of Q is realized in one of the directions $(0, 1)$, $(1, 0)$, $(1, 1)$, and $(-1, 1)$. The lattice width of Q in direction $q \in \mathbb{Z}^2$ is $w_L(q, Q) := \max_{x,y \in Q} q(x - y)$.

Figure 5

In case (i) of Theorem 2 (see Figure 5) $x = u$ follows from computing the area of Q in two ways. Similarity of triangles implies $z : x = (L - x) : y$. We get

$$\begin{aligned} w_L((1,0),\, Q) &= L + y - x\,, & w_L((0,1),\, Q) &= L + z - x\,, \\ w_L((1,1),\, Q) &= 2L + y - 2x - z\,, & w_L((1,1),\, Q) &= 2L + z - 2x - y\;. \end{aligned} \tag{9}$$

Then

$$w_L(Q) = \min\,(L + y - x,\, L - z - x) = L - x + \min\left(y,\, \frac{(L-x)x}{y}\right)$$

and a simple analysis shows

$$w_L(Q) \le \frac{1+\sqrt{2}}{2} L \approx 1.207\ldots\; L\;.$$

In case (ii) see Figure 6.

Figure 6

For the left-hand-side hexagon note that the position of S_3 does not influence the width of Q as long as S_3 cuts off two opposite vertices of the parallelogram $S_1 \cap S_2$. So we may place S_3 so as to contain the isosceles and right angle triangle of Figure 6. Reflecting inwards the three small triangles and comparing areas gives

$$\frac{1}{2}m_1L + \frac{1}{2}m_2L + \frac{1}{2}m_3\sqrt{2}L \le \frac{1}{2}L^2$$

implying

$$\min(m_1,\, m_2,\, \sqrt{2}m_3) \le \frac{1}{3}L\,.$$

Further, $w_L((1,0),\, Q) = L + m_2$, $w_L((0,1),\, Q) = L + m_1$, and $w_L((-1,1),\, Q) = L + \sqrt{2}m_3$. So $w_L(Q) \le \frac{4}{3}L$.

For the other hexagon of Figure 6 the computations in (9) can easily be applied. □

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