A NEW GENERALIZATION OF THE ERDŐS—KO—RADO THEOREM

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Dedicated to Paul Erdős on his seventieth birthday

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Let $\mathcal{F}$ be a family of k -subsets of an n -set. Let s be a fixed integer satisfying $k \leq s \leq 3k$. Suppose that for $F_1, F_2, F_3 \in \mathcal{F}$ $|F_1 \cup F_2 \cup F_3| \leq s$ implies $F_1 \cap F_2 \cap F_3 \neq \emptyset$. Katona asked what is the maximum cardinality, $f(n, k, s)$ of such a system. The Erdős—Ko—Rado theorem implies $f(n, k, s) = \binom{n-1}{k-1}$ for $s = 3k$ and $n \geq 2k$. In this paper we show that $f(n, k, s) = \binom{n-1}{k-1}$ holds for $n > n_0(k)$ if and only if $s \geq 2k$.

Equality holds only if every member of $\mathcal{F}$ contains a fixed element of the underlying set.

Further we solve the problem for $k = 3, s = 5, n \geq 3000$. This result sharpens a theorem of Bollobás.

1. Introduction

The simplest version of the Erdős—Ko—Rado theorem is the following

Theorem 1. [4] Let $\mathcal{F}$ be a collection of k -element subsets of an n -set X . Suppose $F \cap F' \neq \emptyset$ for $F, F' \in \mathcal{F}$. Then for $n > 2k$

$$(1) \quad |\mathcal{F}| \leq \binom{n-1}{k-1},$$

and equality holds iff for some $x \in X$ we have

$$(2) \quad \mathcal{F} = \{F \subset X \mid |F| = k, x \in F\}.$$

In Frankl [5] the following is proven

Theorem 2. Let $\mathcal{F}$ be a collection of k -element subsets of an n -set X , and let $t \geq 2$. Suppose that, for every $F_1, F_2, \dots, F_t \in \mathcal{F}$, $F_1 \cap \dots \cap F_t \neq \emptyset$ holds. Then for $n > (t/t-1)k$ (1) holds. Equality is possible only for $\mathcal{F}$ satisfying (2).

Katona raised the following problem, concerning the case $t = 3$ of Theorem 2. What happens if, for some integer s , we require $F_1 \cap F_2 \cap F_3 \neq \emptyset$ only for triples satisfying $|F_1 \cup F_2 \cup F_3| \leq s$? For which values of s does the condition entail (1)? In this paper we investigate this problem for $n \geq n_0(k)$, and show that (1) holds whenever $s \geq 2k$.

2. Results

Theorem 3. Let $\mathcal{F}$ be a collection of k -element subsets of the n -set X . Suppose that for any $F_1, F_2, F_3 \in \mathcal{F}$, satisfying $|F_1 \cup F_2 \cup F_3| \leq 2k$ $F_1 \cap F_2 \cap F_3 \neq \emptyset$ holds. Then there is a number $n_0(k)$ such that, for $n > n_0(k)$

$$|\mathcal{F}| \leq \binom{n-1}{k-1},$$

and equality holds only if $\mathcal{F}$ is a family consisting of all the k -subsets containing a fixed element. Moreover $n_0(3)=5$, $n_0(k) \leq k^2 + 3k$.

It is somewhat surprising that the extremal family is unchanged in the range $2k \leq s \leq 3k$.

However for $s < 2k$ the situation is completely different, as it is shown by the following construction.

Let us consider a partition of X into k sets $X_1, \dots, X_k$ with $\left\lfloor \frac{n}{k} \right\rfloor \leq |X_i| \leq \left\lceil \frac{n}{k} \right\rceil + 1$.

Let us define

$$(3) \quad \mathcal{G} = \{G \subset X \mid |G \cap X_i| = 1 \text{ for } 1 \leq i \leq k\}.$$

Suppose now $G_1 \cap G_2 \cap G_3 = \emptyset$ for some $G_1, G_2, G_3 \in \mathcal{G}$. Then obviously for every $1 \leq i \leq k$ we have

$$|(G_1 \cup G_2 \cup G_3) \cap X_i| \geq 2.$$

From this we immediately obtain $|G_1 \cup G_2 \cup G_3| \geq 2k$, in other words $|G_1 \cup G_2 \cup G_3| \leq 2k-1$ implies $G_1 \cap G_2 \cap G_3 \neq \emptyset$, i.e. $\mathcal{G}$ satisfies the condition of Katona. $|\mathcal{G}| \leq \left\lceil \frac{n}{k} \right\rceil^k$ which is of greater order of magnitude than $\binom{n-1}{k-1}$.

Conjecture. Let $\mathcal{F}$ be a family of k -subsets of X , $|X|=n$. Suppose $F_1, F_2, F_3 \in \mathcal{F}$ $|F_1 \cup F_2 \cup F_3| \leq 2k-1$ implies $F_1 \cap F_2 \cap F_3 \neq \emptyset$. Then for $n \geq n_0(k)$ and $\mathcal{G}$ defined above

$$|\mathcal{F}| \leq |\mathcal{G}|,$$

with equality iff $\mathcal{F} = \mathcal{G}$.

Theorem 4. If $k=3$ and $n \geq 3000$ then $f(n, 3, 5) = \left\lfloor \frac{n}{3} \right\rfloor \left\lfloor \frac{n+1}{3} \right\rfloor \left\lfloor \frac{n+2}{3} \right\rfloor$.

This result is a sharpening of the following theorem.

Theorem 5. (Bollobás [1]) Let $\mathcal{F}$ be a family of 3-subsets of X , $|X|=n$. Suppose that for $F_1, F_2, F_3 \in \mathcal{F}$ we have $F_1 \Delta F_2 \subset F_3$ (Δ denotes the symmetric difference). Then $|\mathcal{F}| \leq |\mathcal{G}|$ with equality holding only if $\mathcal{F}$ is isomorphic to $\mathcal{G}$.

Thus Bollobás excludes the configuration when F_1, F_2, F_3 are three different 3-subset of a 4-set, while Theorem 4 permits it. However Bollobás's result holds for every n while we assume $n \geq 3000$, and our theorem is definitely not true for $n \leq 10$.

As for $k=3, s \leq 4$, trivially $f(n, 3, s) = \binom{n}{3}$ holds, therefore Katona's problem is solved for $k=3$, except when $s=5$, $n < 3000$.

3. The proof of Theorem 3

When $k=2$, $\mathcal{F}$ is a simple graph containing no triangles or path of length 3, so it is the union of vertex disjoint stars, thus Theorem 3 is true. From now on assume that $k \geq 3$.

We proceed in a similar way as in Frankl [6]. First we prove that (1) holds asymptotically. Let $m(n, k, 1)$ denote the maximum number of k -subsets of an n -set, such that no two intersect in a singleton.

Then we have:

Lemma 1. *If $\mathcal{F}$ satisfies the conditions of Theorem 3, then*

$$(4) \quad |\mathcal{F}| \leq \binom{n}{k-1} + m(n, k, 1).$$

Proof. Let $\mathcal{F}_0$ be the family of those subsets of $\mathcal{F}$ which contain a $(k-1)$ -subset not contained in any other member of $\mathcal{F}$, i.e. $\mathcal{F}_0 = \{F \in \mathcal{F} \mid \exists G \subset F, |G|=k-1, G \subset F' \in \mathcal{F} \text{ implies } F'=F\}$, and define $\mathcal{F}_1 = \mathcal{F} - \mathcal{F}_0$.

Clearly $|\mathcal{F}_0| \leq \binom{n}{k-1}$. Hence it suffices to prove $|\mathcal{F}_1| \leq m(n, k, 1)$. Suppose the contrary, then we can find $F_1, F_2 \in \mathcal{F}_1$ such that $|F_1 \cap F_2| = 1$. Let $F_1 \cap F_2 = \{x\}$. As $F_1 \notin \mathcal{F}_0$ there is an $F_3 \in \mathcal{F}, F_1 \neq F_3$ such that $(F_1 - \{x\}) \subset F_3$. But in this case $F_1 \cap F_2 \cap F_3 = \emptyset$ and $|F_1 \cup F_2 \cup F_3| \leq |F_1 \cup F_2| + 1 = 2k$, a contradiction. ■

The problem of determining $m(n, k, 1)$ was raised by Erdős and Sós (see [2]), who determined $m(n, 3, 1)$, in particular they proved $m(n, 3, 1) \leq n$, and conjectured $m(n, k, 1) = \binom{n-2}{k-2}$ for $n \geq 2k$. This was proved by Frankl [7] for $n > n_0(k)$. Since

$$\binom{n}{k-1} = \binom{n-1}{k-1} + \binom{n-1}{k-2}, \text{ Lemma 1 yields}$$

Corollary 1. *If $\mathcal{F}$ satisfies the conditions of Theorem 3, then for $n > n_0(k)$*

$$(5) \quad |\mathcal{F}| \leq \binom{n-1}{k-1} + \binom{n-1}{k-2} + \binom{n-2}{k-2} < (1 + 3k/n) \binom{n-1}{k-1}.$$

In the proof of Lemma 1 we used only:

Proposition 0. *If $F_1, F_2 \in \mathcal{F}$ and $F_1 \cap F_2 = \{x\}$ then there are no sets F'_1 or F'_2 in $\mathcal{F}$ satisfying $(F_1 - \{x\}) \subset F'_1$ or $(F_2 - \{x\}) \subset F'_2$. ■*

For $x \in X$ let $\mathcal{D}(x)$ denote the family of sets $F \in \mathcal{F}$ with $x \in F$, and $\mathcal{D}_0(x)$ the family of sets $F \in \mathcal{F}$ with $x \in F$ such that $F - \{x\}$ is not contained in any other $F' \in \mathcal{F}$. Let further $|\mathcal{D}(x)| = d(x)$, $|\mathcal{D}_0(x)| = d_0(x)$. Clearly we have

$$(6) \quad \sum_{x \in X} d(x) = k|\mathcal{F}|,$$

$$(7) \quad \sum_{x \in X} d_0(x) \leq \binom{n}{k-1}.$$

In view of Proposition 0 we have

Proposition 1. Suppose $F_1, F_2 \in \mathcal{F}$ and $F_1 \cap F_2 = \{x\}$. Then $F_1, F_2 \in \mathcal{D}_0(x)$. ■

Let us set $\mathcal{A}(x) = \{F - \{x\} \mid F \in (\mathcal{D}(x) - \mathcal{D}_0(x))\} = \{F - \{x\} \mid x \in F \in \mathcal{F} \text{ and } \exists F' \in \mathcal{F} \text{ with } (F - \{x\}) \subset F'\}$. Then Proposition 1 yields:

Proposition 2. For $A, A' \in \mathcal{A}(x)$ we have $A \cap A' \neq \emptyset$. ■

We will use the following theorem of Hilton and Milner:

Theorem 6. [9] Let $\mathcal{A}$ be a collection of r -element subsets of an n -set, $n \geq 2r$. Suppose that $A \cap A' \neq \emptyset$ for $A, A' \in \mathcal{A}$ and $|\mathcal{A}| > \binom{n-1}{r-1} - \binom{n-r-1}{r-1} + 1$. Then there exists an element y such that $y \in A$ for every $A \in \mathcal{A}$. ■

Proposition 3. If $r \geq 2$, $n \geq 2r$, then

$$(8) \quad \binom{n-1}{r-1} - \binom{n-r-1}{r-1} + 1 \leq r \binom{n-2}{r-2} + 1.$$

Proof. It follows from

$$\binom{n-1}{r-1} - \binom{n-r-1}{r-1} = \sum_{i=0}^{r-1} \binom{n-2-i}{r-2}. \quad \blacksquare$$

Proposition 4. If $|\mathcal{A}(x)| > k \binom{n-3}{k-3}$ then there exists $y \in (X - x)$ such that $y \in F$ for every $F \in \mathcal{D}(x)$.

Proof. In view of Propositions 2 and 3 we can find $y \in (X - x)$ such that $y \in A$ for every $A \in \mathcal{A}(x)$. Suppose that for some $F \in \mathcal{D}_0(x)$ we have $y \notin F$. In view of Proposition 1 for every $A \in \mathcal{A}(x)$ we have $A \cap F \neq \emptyset$, yielding $|\mathcal{A}(x)| \leq k \binom{n-3}{k-3}$, a contradiction. ■

Call the point $x \in X$ *good* if there exists a $y \neq x$ such that $x \in F \in \mathcal{F}$ entails $y \in F$. If x is good then fix one such y and denote it by $f(x)$.

Corollary 2. If $|\mathcal{A}(x)| > k \binom{n-3}{k-3}$ then x is good. ■

We assume from now on that $|\mathcal{F}| \leq \binom{n-1}{k-1}$ and that there is no vertex $z \in X$ which is contained in every member of $\mathcal{F}$.

Lemma 2. If x is good then $d(x) \leq \binom{n-2}{k-2} - (k-1) \binom{n-k-2}{k-3}$.

Proof. By the indirect assumption there exists $F_0 \in \mathcal{F}$ with $f(x) \notin F_0$. As x is good and $f(x) \notin F_0$ we have $x \notin F_0$. Let us consider k -subsets of the following form: $F = G \cup \{y\} \cup \{x, f(x)\}$, where $y \in F_0$, $G \subset X - (F_0 \cup \{x, f(x)\})$, $|G| = k-3$. The total number of such k -sets is $k \binom{n-k-2}{k-3}$. As for given G_0 and $y_1, y_2 \in F_0$ the intersection

of the three sets $F_0, G_0 \cup \{y_1, x, f(x)\}, G_0 \cup \{y_2, x, f(x)\}$ is empty and their union of cardinality less than $2k$, so at most one set of the form $G_0 \cup \{y, x, f(x)\}$ ($y \in F_0$) belongs to $\mathcal{F}$. Consequently, at most $\binom{n-k-2}{k-3}$ sets of the form $\{G \cup \{y, x, f(x)\} \mid y \in F_0, G \subset (X - F_0 \cup \{x, f(x)\}), |G| = k-3\}$ belong to $\mathcal{F}$. This means that at least $(k-1)\binom{n-k-2}{k-3}$ sets are missing from the $\binom{n-2}{k-2}$ possible k -sets containing $\{x, f(x)\}$. ■

Lemma 3. *If $n > k^2 - k$ then there exists at least one good vertex x .*

Proof. Suppose the contrary then using (6), (7) and Corollary 2 we deduce

$$k \binom{n-1}{k-1} \leq k|\mathcal{F}| = \sum_{x \in X} d(x) = \sum_{x \in X} d_0(x) + \sum_{x \in X} |\mathcal{A}(x)| \leq \binom{n}{k-1} + nk \binom{n-3}{k-3}.$$

It is easy to see that the right hand side is less than $k \binom{n-1}{k-1}$ for $n > k^2 - k$, a contradiction. ■

We prove Theorem 3 for $k=3$, $n \geq 5 = n_0(3)$. For $n=5, 6$ it follows from Theorem 2. We apply induction on n .

Let $\mathcal{F}$ be a family satisfying the assumptions but not the statement. As $n \geq 7$, by Lemmas 2 and 3, we can find $x \in X$ with $d(x) < (n-2)$. Then $\mathcal{F} - \mathcal{D}(x)$ is a family satisfying the assumptions on $X - \{x\}$. We may use the induction hypothesis $|\mathcal{F} - \mathcal{D}(x)| \leq \binom{n-2}{2}$, yielding $|\mathcal{F}| < \binom{n-2}{2} + (n-2) = \binom{n-1}{2}$ which concludes the proof.

From now on we assume that $k \geq 4$. Let us suppose $n > k^2 - k$. Suppose the statement of the theorem is false for $\mathcal{F}$. Then by $n > k^2 - k$ there exists $x \in X$ with $d(x) \leq \binom{n-2}{k-2} - (k-1) \binom{n-k-2}{k-3}$. Let $X_1 = X - \{x\}$, and $\mathcal{F}_1 = \mathcal{F} - \mathcal{D}(x)$. If $X_i, \mathcal{F}_i$ are defined with $|\mathcal{F}_i| > \binom{|X_i|-1}{k-1}$ then let $x \in X_i$ with $d(x) \leq \binom{|X_i|-2}{k-2} - (k-1) \cdot \binom{|X_i|-k-2}{k-3}$ (such a vertex exists certainly for $|X_i| > k^2 - k$.)

Let $X_{i+1} = X_i - \{x\}$, $\mathcal{F}_{i+1} = \mathcal{F}_i - \mathcal{D}(x)$. Let j be the index for which $|X_j| = k^2 - k$, i.e., $j = n - k^2 + k$. Then we have

$$(9) \quad |\mathcal{F}_j| \leq |\mathcal{F}| - \sum_{i=0}^{j-1} \left(\binom{n-2-i}{k-2} - (k-1) \binom{n-k-2-i}{k-3} \right) \leq \binom{n-1}{k-1} - \sum_{i=0}^{j-1} \binom{n-2-i}{k-2} + (k-1) \sum_{i=0}^{j-1} \binom{n-k-2-i}{k-3} = \binom{k^2-k-1}{k-1} + (k-1) \left(\binom{n-k-1}{k-2} - \binom{k^2-2k-1}{k-2} \right).$$

On the other hand $\mathcal{F}_j$ is a family of k -subsets of the (k^2-k) -element set X_j , thus by Lemma 1

$$(10) \quad |\mathcal{F}_j| \leq \binom{k^2-k}{k-1} + m(k^2-k, k, 1).$$

Proposition 5. For $n > 2k$ we have

$$m(n, k, 1) \leq \frac{n}{k} \binom{n-2}{k-2}.$$

Proof. Let $\mathcal{G}$ be a family of k -subsets of an n -set which does not contain two members intersecting in a singleton. Then for every vertex x , $\mathcal{G}_x = \{G - \{x\} : x \in G \in \mathcal{G}\}$ is an intersecting family of $(k-1)$ -subsets of an $(n-1)$ -set. Thus by the Erdős—Ko—Rado theorem (Theorem 1) we have $|\mathcal{G}_x| \leq \binom{n-2}{k-2}$. Therefore

$$|\mathcal{G}| = \frac{1}{k} \sum_x |\mathcal{G}_x| \leq \frac{n}{k} \binom{n-2}{k-2}. \quad \blacksquare$$

Combining (10) with Proposition 5, we obtain

$$(11) \quad |\mathcal{F}_j| \leq \binom{k^2-k}{k-1} + (k-1) \binom{k^2-k-2}{k-2}.$$

However, for $n \geq k^2 + 3k$, (11) contradicts (9), which concludes the proof of Theorem 3.

Remark 1. Proposition 4 remains true for $|\mathcal{A}(x)| > \binom{n-2}{k-2} - \binom{n-k-1}{k-2} + 1$. Using this one can prove Lemma 3 for $n > k^2/(1.5 \log k)$ and in this way the upper bound $n_0(k) \leq k^2 + 3k$ can be improved to $n_0(k) < k^2/\log k$. But it is still far from the real value of $n_0(k)$ which we conjecture to be $\lceil 3k/2 \rceil$. We can prove this for $k=4, 5$.

4. The proof of Theorem 4.

With the family $\mathcal{F}$ let us associate the graph $\mathcal{A}$ whose vertex set is X and whose edges are all the 2-sets which are contained in some $F \in \mathcal{F}$.

Let us recall now a result of Erdős [3]. For simplicity we state it only for a special case.

Theorem 7. [3] Let $\mathcal{F}$ be a family of 3-subsets of X , $|X|=n$. Suppose that $\mathcal{A}$ contains no complete subgraph on 4 vertices. Then for $\mathcal{F}$ the assertion of Theorem 4 holds.

Let s be the greatest number for which $\mathcal{A}$ contains a complete subgraph on s vertices.

If $s=3$ then Theorem 7 yields the statement of our theorem. For $s \geq 4$ we will proceed in a similar way as with the proof of Theorem 3. Let $t = \min(s, 5)$. Let $x_1, \dots, x_t$ be the vertices of a complete subgraph of $\mathcal{A}$. By Turán's theorem [9] we have for $t=s=4$

$$(12) \quad |\mathcal{A}| \leq \frac{3}{8} n^2,$$

Let $\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3$ be the collection of members B of $\mathcal{F}$ for which $|B \cap \{x_1, \dots, x_t\}|$

$=1, 2, 3$, respectively. Obviously, we have

$$(13) \quad |\mathcal{B}_3| \equiv \binom{t}{3},$$

and

$$(14) \quad |\mathcal{B}_2| \equiv \binom{t}{2}(n-t).$$

For $1 \leq i < j \leq t$, let us choose $z_{i,j} \in X$ such that $\{x_i, x_j, z_{i,j}\} \in F$. This is possible since $\{x_i, x_j\} \in \mathcal{A}$. Let Z be the set of different $z_{i,j}$'s. Of course $|Z| \leq \binom{t}{2}$.

Proposition 6. *If for $1 \leq i < j \leq t$, and for $y_1, y_2 \in X$ both $\{x_i, y_1, y_2\}$ and $\{x_j, y_1, y_2\}$ belong to $\mathcal{F}$, then either y_1 or y_2 belongs to Z .*

Proof. Let us write $z = z_{i,j}$. By definition $\{x_i, x_j, z\} \in \mathcal{F}$. Then $|\{x_i, x_j, z\} \cup \{x_i, y_1, y_2\} \cup \{x_j, y_1, y_2\}| \leq 5$, consequently the intersection of the 3 sets is non-empty, i.e., $z = y_1$ or $z = y_2$, as desired. ■

Proposition 7.

$$(15) \quad |\mathcal{B}_1| < 2 \binom{t}{2}(n-t) + \frac{t-1}{8} n^2.$$

Proof. Our first claim is that the first term is an upper bound for the number of $F \in \mathcal{F}$ with $|F \cap \{x_1, \dots, x_t\}| = 1$, $F \cap Z \neq \emptyset$. For $z \in Z$ let $m(z)$ denote the multiplicity of Z , i.e. the number of pairs (i, j) , $1 \leq i < j \leq t$ with $z = z_{i,j}$. For $z \in Z$ and $y \in X - \{x_1, \dots, x_t\}$ let $D(z, y)$ denote the set of x_i , $1 \leq i \leq t$ such that $\{z, y, x_i\} \in \mathcal{F}$. If $y \notin Z$ then by Proposition 6 for $x_i, x_j \in D(z, y)$ we have $z = z_{i,j}$. If $y \in Z$ then the only other possibility is $y = z_{i,j}$. Thus for $y \notin Z$ we have $m(z) \leq \binom{|D(z, y)|}{2}$, in particular $2m(z) \leq |D(z, y)|$ holds. Similarly, if $y \in Z$ then $2m(z) + 2m(y) \leq |D(z, y)|$. Summing up these inequalities for all pairs $z \in Z$, $y \in X - \{x_1, \dots, x_t\}$, considering the pairs with $y \in Z$ only once and taking into consideration $\sum_{z \in Z} m(z) = \binom{t}{2}$ we obtain our first claim. In view of Proposition 6 and (12) the second term is an upper bound for $|\mathcal{A}|$, which is at least the number of $F \in \mathcal{F}$ with $|F \cap \{x_1, \dots, x_t\}| = 1$, $F \cap Z \neq \emptyset$. ■

Now summing (13), (14) and (15) we obtain

$$(16) \quad \min_{1 \leq i \leq t} d(x_i) \leq \frac{1}{t} \sum_{i=1}^t d(x_i) = \frac{1}{t} (3|\mathcal{B}_3| + 2|\mathcal{B}_2| + |\mathcal{B}_1|) \\ \leq \begin{cases} \frac{3}{32} n^2 + 6n - 21 & \text{for } t = 4 \\ \frac{1}{10} n^2 + 8n - 37 & \text{for } t = 5 \end{cases} \leq \frac{1}{10} n^2 + 8n - 37 \quad (\text{if } n \geq 8).$$

Suppose now that $n > 3000$, $|\mathcal{F}| \cong \left\lfloor \frac{n}{3} \right\rfloor \left\lfloor \frac{n+1}{3} \right\rfloor \left\lfloor \frac{n+2}{3} \right\rfloor$ and the theorem is false. By (16) we can take an $x_1 \in X$, $X_1 = X - \{x_1\}$ and $\mathcal{F}_1 = \{F \in \mathcal{F} | x_1 \notin F\}$ such that $d(x_1) \cong \frac{1}{10}n^2 + 8n - 37$. Then, in view of (16), $|\mathcal{F}_1| > \left\lfloor \frac{n-1}{3} \right\rfloor \left\lfloor \frac{n}{3} \right\rfloor \left\lfloor \frac{n+1}{3} \right\rfloor = |\mathcal{G}_{n-1}|$ and we can argue in the same way for $\mathcal{F}_1$ as we did for $\mathcal{F}$. Let q be the first integer with $|X_q| \cong 750$. Then $q = n - 750$. For the cardinality of $\mathcal{F}_q$ we deduce

$$\begin{aligned} (17) \quad |\mathcal{F}_q| &> |\mathcal{F}| - \sum_{i=0}^{q-1} \left(\frac{1}{10}(n-i)^2 + 8(n-i) - 37 \right) \\ &= |\mathcal{F}| - \frac{1}{10} \frac{n(n+1)(2n+1)}{6} - 4n(n+1) + 37n + \frac{1}{10} \frac{750 \cdot 751 \cdot 1501}{6} \\ &\quad + 4 \cdot 750 \cdot 751 - 37 \cdot 750. \end{aligned}$$

Now using the assumption $|\mathcal{F}| \cong \frac{1}{27}(n^3 - 3n - 2)$ we obtain from (17), for $n > 3000$, $|\mathcal{F}_q| > \frac{1}{270}n^3 - 4.1n^2 + 16\,000\,000 > \binom{750}{3}$, a contradiction, proving the theorem.

5. Concluding remarks

Remark 2. Theorem 4 is not only a sharpening of Theorem 5, but the proof is entirely new.

Remark 3. The problems considered in this paper belong to the so-called Turán-type problems, i.e. what is the maximum number of k -subsets of an n -set if it contains no sub-system isomorphic to one member of a set of k -graphs $\{\mathcal{H}_1, \mathcal{H}_2, \dots, \mathcal{H}_q\}$. This maximum is usually denoted by $\text{ext}(n, \{\mathcal{H}_1, \mathcal{H}_2, \dots, \mathcal{H}_q\})$.

Let us define $\mathcal{H}_1 = \{\{x_1, x_2, \dots, x_k\}, \{x_1, x_2, \dots, x_{k-1}, x_{k+1}\}, \{x_{k+1}, x_{k+2}, \dots, x_{2k}\}\}$, $\mathcal{H}_2 = \{\{x_1, x_2, \dots, x_k\}, \{x_1, x_2, \dots, x_{k-1}, x_{k+1}\}, \{x_k, x_{k+1}, \dots, x_{2k-1}\}\}$. In this terminology we proved (Theorem 4) for $k=3$ $\text{ext}(n, \{\mathcal{H}_2\}) = \left\lfloor \frac{n}{3} \right\rfloor \left\lfloor \frac{n+1}{3} \right\rfloor \left\lfloor \frac{n+2}{3} \right\rfloor$. Moreover, the proof of Theorem 3 yields for $n > n_0(k)$ the stronger result $\text{ext}(n, \{\mathcal{H}_1, \mathcal{H}_2\}) = \binom{n-1}{k-1}$.

Refining the argument we could even obtain

Theorem 8. For $n > n_1(k)$ we have $\text{ext}(n, \{\mathcal{H}_1\}) = \binom{n-1}{k-1}$.

Finally a special case of a result of the first author gives

Theorem 9. [8] Let $\mathcal{H} = \{H_1, H_2, H_3\}$ be an arbitrary k -graph satisfying $|H_1 \cup H_2 \cup H_3| \cong 2k$, $H_1 \cap H_2 \cap H_3 \neq \emptyset$. Then for every n , $\text{ext}(n, \{\mathcal{H}\}) < 3en^{k-1}$.

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