

NEW ASYMPTOTICS FOR BIPARTITE TURÁN NUMBERS

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ABSTRACT. An algebraic construction implies $\lim_{n \rightarrow \infty} \text{ex}(n, K_{2,t+1})n^{-3/2} = \sqrt{t}/2$.

1. THE TURÁN PROBLEM

Given a graph F , what is $\text{ex}(n, F)$, the maximum number of edges of a graph with n vertices not containing F as a subgraph? Until now, the only asymptotic for a bipartite graph which is not a forest, $\text{ex}(n, C_4) = \frac{1}{2}(1 + o(1))n^{3/2}$, is due to Erdős, Rényi and T. Sós [ERS] and (simultaneously and independently) to Brown [B].

Theorem. For any fixed $t \geq 1$ $\text{ex}(n, K_{2,t+1}) = \frac{1}{2}\sqrt{tn}^{3/2} + O(n^{4/3})$.

Let G be a graph on n vertices with e edges such that any two vertices have at most t common neighbors. The inequality $\sum_{x \in V} \binom{d(x)}{2} \leq t \binom{n}{2}$ (Kővári, T. Sós, Turán [KST]) implies the upper bound $e < \frac{1}{2}\sqrt{tn}^{3/2} + (n/4)$. To prove the Theorem we obtain a matching lower bound from a construction closely related to the examples from [ERS] and [B], and inspired by an example of Hyltén-Cavallius [H] and Mörs [M] given for Zarankiewicz's problem $z(n, n, 2, t+1)$ (see later in Section 3). The topic is so short of constructions that about 20 years ago, as a first step, Erdős [E67, E75] even proposed the problem whether $\lim_t(\liminf_n \text{ex}(n, K_{2,t+1})n^{-3/2})$ goes to ∞ as $t \rightarrow \infty$. For a survey see Bollobás' book [Bo].

2. A LARGE GRAPH WITH NO $K_{2,t+1}$

Let q be a prime power such that $(q-1)/t$ is an integer. We will construct a $K_{2,t+1}$ -free graph G on $n = (q^2 - 1)/t$ vertices such that every vertex has degree q or $q-1$. Then G has more than $(1/2)\sqrt{tn}^{3/2} - (n/2)$ edges. The lower bound for the Turán number for all n then follows from the fact for every sufficiently large n there exists a prime q satisfying $q \equiv 1 \pmod{t}$ and $\sqrt{nt} - n^{1/3} < q < \sqrt{nt}$ (see [HI]).

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Let \mathbf{F} be the q -element finite field, and let h be an element of order t , i.e., $h^t = 1$ and the set $H = \{1, h, h^2, \dots, h^{t-1}\}$ form a t -element subgroup of $\mathbf{F} \setminus \{0\}$. We say that $(a, b) \in \mathbf{F} \times \mathbf{F}$, $(a, b) \neq (0, 0)$ is equivalent to (a', b') , in notation $(a, b) \sim (a', b')$, if there exists some $h^\alpha \in H$ such that $a' = h^\alpha a$ and $b' = h^\alpha b$. The elements of the vertex set V of G are the t -element equivalence classes of $\mathbf{F} \times \mathbf{F} \setminus (0, 0)$. The class represented by (a, b) is denoted by $\langle a, b \rangle$. Two (distinct) classes $\langle a, b \rangle$ and $\langle x, y \rangle$ are joined by an edge in G if $ax + by \in H$. This relation is symmetric, and compatible to the equivalence classes, i.e., $ax + by \in H$, $(a, b) \sim (a', b')$, and $(x, y) \sim (x', y')$ imply $a'x' + b'y' \in H$.

For any given $(a, b) \in \mathbf{F} \times \mathbf{F} \setminus (0, 0)$ (say, $b \neq 0$) and for any given x and h^α , the equation $ax + by = h^\alpha$ has a unique solution for y . This implies that there are exactly tq solutions (x, y) with $ax + by \in H$. The solutions come in equivalence classes, one of these might coincide with $\langle a, b \rangle$ so the degree of the vertex $\langle a, b \rangle$ in G is either q or $q - 1$.

We claim that G is $K_{2,t+1}$ -free. First we show, that for $(a, b), (a', b') \in \mathbf{F} \times \mathbf{F} \setminus (0, 0)$, $(a, b) \not\sim (a', b')$ the equation system

$$(1) \quad \begin{aligned} ax + by &= h^\alpha \\ a'x + b'y &= h^\beta \end{aligned}$$

has at most one solution $(x, y) \in \mathbf{F} \times \mathbf{F} \setminus (0, 0)$. Indeed, the solution is unique if the determinant $\det \begin{pmatrix} a & b \\ a' & b' \end{pmatrix}$ is not 0. Otherwise, there exists a c such that $a = a'c$ and $b = b'c$. If there exists a solution of (1) at all, then multiplying the second equation by c and subtracting it from the first one we get on the right hand side $h^\alpha - ch^\beta = 0$. Thus $c \in H$, contradicting the fact that (a, b) and (a', b') are not equivalent.

Finally, there are t^2 possibilities for $0 \leq \alpha, \beta < t$ in (1). The set of solutions again form t -element equivalence classes, so there are at most t classes $\langle x, y \rangle$ joint simultaneously to $\langle a, b \rangle$ and $\langle a', b' \rangle$. \square

The sets $N\langle a, b \rangle = \{\langle x, y \rangle : ax + by \in H\}$ almost form a $q + 1$ -uniform, symmetric t -design. This means that, they mutually intersect in exactly t elements except if $(a, b) = (ca', cb')$ holds for some c when they are disjoint. It seems to me that this structure, unfortunately, cannot be extended to a proper t -design.

3. COROLLARIES FOR ZARANKIEWICZ'S PROBLEM

Given m, n, s and t , what is the maximum number, $z = z(m, n, s, t)$, such that there exists a 0–1 matrix with m rows and n columns containing z 1's without a submatrix with s rows and t columns consisting of entirely of 1's. This question has become known as the *problem of Zarankiewicz* [Z]. For a bipartite graph F define the *bipartite Turán number*, $\text{ex}(m, n, F)$, as the maximum number of edges in an F -free bipartite graph with m and n vertices in its color classes. Considering the adjacency matrix of a $K_{s,t}$ -free graph on n vertices as a matrix of a

bipartite graph on $n + n$ vertices (cf. [Bo] p. 310) we get

$$(2) \quad 2\text{ex}(n, K_{s,t}) \leq \text{ex}(n, n, K_{s,t}) \leq z(n, n, s, t).$$

The determination of $z(m, n, s, t)$ is equivalent to a unidirectional Turán problem, when we label the two color classes of F and only those copies of F are forbidden in which the entire first color class is contained in the m -element set and the second color class lies in the n -element set.

It is easy to see that $z(n, n, 2, t + 1) \leq n\sqrt{tn - t + 1/4} + (n/2)$, and it is known that this bound is asymptotically correct, i.e., $\lim_{n \rightarrow \infty} z(n, n, 2, t + 1)n^{-3/2} = \sqrt{t}$ ([KST] for $t = 1$, [H] for $t = 2$ and [M] for all t). Our Theorem and the lower bound in (2) gives

Corollary. *For any fixed $t \geq 1$ $\text{ex}(n, n, K_{2,t+1}) = \sqrt{tn}^{3/2} + O(n^{4/3})$.*

Thus we have a new near optimal construction for $z(n, n, 2, t + 1)$. The gap between the lower and upper bounds in the case $n = (q^2 - 1)/t$ is only $O(\sqrt{n})$.

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