

Maximal τ -critical Linear Hypergraphs ¹

ZOLTÁN FÜREDI ²

*Department of Mathematics, University of Illinois,
1409 W. Green St., Urbana, IL 61801, USA*

E-mail: z-furedi@math.uiuc.edu and

Mathematical Institute of the Hungarian Academy of Sciences

P. O. Box 127, Budapest, 1364 Hungary

E-mail: furedi@math-inst.hu

Abstract

A construction using finite affine geometries is given to show that the maximum number of edges in a τ -critical linear hypergraph is $(1-o(1))\tau^2$. This asymptotically answers a question of Roudneff [14], Aharoni and Ziv [1].

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1 Linear hypergraphs

A *linear hypergraph* \mathcal{H} is an ordered pair $\mathcal{H} = (V, \mathcal{E})$, where $V = V(\mathcal{H})$ is a finite set of vertices and $\mathcal{E} = \mathcal{E}(\mathcal{H})$ is a collection of subsets of V , called *edges*, such that any two edges have at most one common vertex. The *size* of the hypergraph means the number of its edges. A set T is a *cover* (or *transversal*) of \mathcal{H} if it intersects every edge. The minimum cardinality of a cover is the *covering number*, it is denoted by $\tau(\mathcal{H})$.

Let us denote the Desarguesian finite projective plane of order q by $\mathcal{P} = \mathcal{P}_q$. It is obtained from a finite field of size q (cf. [13]). It is a linear hypergraph of vertex set of size $q^2 + q + 1$ (called points) and the same number of hyperedges (lines). It is also intersecting (i.e., $L \cap L' \neq \emptyset$ for every two lines $L, L' \in \mathcal{E}(\mathcal{P})$) thus every line is a cover. It is well-known (and easy) that $\tau(\mathcal{P}_q) = q + 1$.

Call a cover $B \subset V(\mathcal{P}_q)$ *non-trivial* or a *blocking set* if it contains no line, i.e., $L \cap B \neq \emptyset$ and $L \not\subset B$ hold for every $L \in \mathcal{E}(\mathcal{P})$. We are going to use the following result of Blokhuis [4]. The size of a blocking set B of a Desarguesian \mathcal{P}_q with q an odd prime is at least

$$|B| \geq \frac{1}{2}(3q + 3). \quad (1)$$

The Desarguesian affine plane, \mathcal{A}_q , of order q is again a linear hypergraph on q^2 vertices and with $q^2 + q$ q -element hyperedges (for details see [13]). Its edge-set can be partitioned into $q + 1$ *parallel classes*, $\mathcal{E}(\mathcal{A}_q) = \mathcal{L}_1 \cup \dots \cup \mathcal{L}_{q+1}$ such that $\mathcal{L}_i = \{L_i^1, L_i^2, \dots, L_i^q\}$ and these form a q -partition of the underlying set $V(\mathcal{A}_q)$ into q -element parts. One has $L_i^\alpha \cap L_j^\beta \neq \emptyset$ for $1 \leq i < j \leq q + 1$ and for arbitrary $1 \leq \alpha, \beta \leq q$. Take a line L_i^α and choose an element from each parallel line $x^\beta \in L_i^\beta$ ($\beta \neq \alpha$), then the obtained $(2q - 1)$ -element set forms a cover of the affine plane. Jamison [12] and (independently) Brouwer and Schrijver [7] proved that for the Desarguesian affine plane \mathcal{A}_q with a prime order q these are among the smallest covers, i.e.,

$$\tau(\mathcal{A}_q) = 2q - 1. \quad (2)$$

The affine plane can be obtained from the projective plane \mathcal{P}_q by deleting the vertices of a line L_0 from its vertex set, and by restricting the remaining $q^2 + q$ lines into $V(\mathcal{P}_q) \setminus L_0$, i.e., deleting one vertex (the one in $L \cap L_0$) from each of the remaining lines $L \in \mathcal{E}(\mathcal{P}_q) \setminus \{L_0\}$. In this case L_0 is called the *line of infinity* of \mathcal{A}_q .

Proofs and further results about covers and especially about blocking sets in projective geometries can be found in the excellent surveys by Blokhuis [5] and Szőnyi [16].

2 τ -critical hypergraphs

A hypergraph is *τ -critical* if omitting one edge always reduces its covering number. Erdős, Hajnal and Moon [9] proved that any τ -critical *graph* has at most $\binom{\tau+1}{2}$ edges, with equality

holding only for the complete graph on $\tau + 1$ vertices. As a generalization Roudneff [14] as well as Aharoni and Ziv [1] conjectured the same upper estimate for every τ -critical linear hypergraph. Let $\ell(\tau)$ denote the maximum size of a τ -critical linear hypergraph. It is easy to see that the maximum degree of such a hypergraph is at most τ and there are sets of size $\tau - 1$ that cover all but one of the edges, we obtain $\ell(\tau) \leq \tau^2 - \tau + 1$. Sudakov [15] proved the conjecture for $\tau \leq 5$ and obtained an upper bound $\ell(\tau) \leq \tau^2 - 3\tau + 5$ for $\tau \geq 5$.

The aim of this note is give an example showing that τ -critical linear hypergraphs are much closer to finite geometries than to graphs. The construction given in the next Section yields

THEOREM 1: $\ell(\tau) \geq \tau^2 - O(\tau^{8/5})$.

For example, (starting with the affine geometry of order 7) we have $\ell(10) \geq 56 > \binom{11}{2}$.

3 Construction

Using affine geometries we are going to define a linear, τ -critical hypergraph \mathcal{H} . Our construction is inspired by an example of Blokhuis [3] (given for a different problem) which was inspired by a result of Drake [8], etc.

Let q be an odd prime, $q \geq 7$ and let r be the largest integer with $\binom{r}{2} < q$, and let $m := \binom{r}{2}$. We have $r := \lfloor \frac{1}{2}(1 + \sqrt{8q + 1}) \rfloor = \sqrt{2q} + O(1)$, and $4 \leq r < m < q$. Let \mathcal{A}_q be the Desarguesian affine geometry of order q with the q^2 -element vertex V_0 and with parallel classes $\mathcal{L}_1, \dots, \mathcal{L}_{q+1}$, $\mathcal{L}_i = \{L_i^1, \dots, L_i^q\}$. Consider $q + 1$ complete graphs of order r with pairwise disjoint vertex sets V_1, \dots, V_{q+1} also disjoint to V_0 . Thus $V := \cup_{0 \leq i \leq q+1} V_i$ has $q^2 + r(q + 1)$ elements. Label the edges of the i 'th complete graph from 1 to m , we obtain $\{E_i^1, E_i^2, \dots, E_i^m\}$. Finally, define the edge-set of \mathcal{H} as

$$\mathcal{E}(\mathcal{H}) := \{L_i^j \cup E_i^j \text{ for } 1 \leq i \leq q+1 \text{ and } 1 \leq j \leq m\} \cup \{L_i^j \text{ for } 1 \leq i \leq q+1 \text{ and } m < j \leq q\}. \quad (3)$$

We claim that \mathcal{H} is a linear, τ -critical hypergraph and the size of the minimum cover is t where this defined as

$$t := q + (r - 2) + (q - m) = q + O(\sqrt{q}). \quad (4)$$

This immediately implies for this value of t that

$$\ell(t) \geq |\mathcal{E}(\mathcal{H})| = q^2 + q = t^2 - O(t^{3/2}). \quad (5)$$

To see the linearity of \mathcal{H} observe that its restriction to V_0 is the affine plane, hence $|E \cap E' \cap V_0| \leq 1$ holds for every pair of edges $E, E' \in \mathcal{E}(\mathcal{H})$. Similarly, $|E \cap E' \cap (V_1 \cup \dots \cup V_{q+1})| \leq 1$. Thus, linearity of \mathcal{H} follows from the fact that if two of its edges E, E' meet in V_0 , $E \cap E' \cap V_0 \neq \emptyset$,

then $E \cap V_0$ and $E' \cap V_0$ belong to distinct parallel classes hence E and E' are disjoint in $V_1 \cup \dots \cup V_{q+1}$. Similarly, if E and E' meet in $V_1 \cup \dots \cup V_{q+1}$ then they are disjoint in V_0 .

It is obvious that $\tau(\mathcal{H}) \leq t$, just consider the cover $T := L_1^1 \cup (V_1 \setminus E_1^1) \cup \{x_1^j : m < j \leq q\}$ where x_1^j is an arbitrary point of L_1^j .

Next we show that $\tau(\mathcal{H}) \geq t$. Consider a cover T and suppose that $|T| \leq t$ thus $|T| < \frac{3}{2}(q+1)$. We are going to show that $|T| \geq t$. The restriction of \mathcal{H} to V_0 is just the affine plane, $\mathcal{H}|_{V_0} \cong \mathcal{A}_q$, thus it is not possible that $T \cap (V_1 \cup \dots \cup V_{q+1}) = \emptyset$. Indeed, otherwise T is a cover for \mathcal{A}_q , too, and then (2) implies that $|T| \geq 2q - 1$, a contradiction. On the other hand, there must be a V_i with $1 \leq i \leq q+1$ such that $V_i \cap T = \emptyset$. Indeed, $|T| < 2q$ implies that $\min_{1 \leq i \leq q+1} |V_i \cap T| \leq 1$. If it is exactly 1, say $|V_1 \cap T| = 1$ and $|V_i \cap T| \geq 1$ for $i > 1$, then $|(V_1 \cup \dots \cup V_{q+1}) \cap T| \geq q+1$ hence $|V_0 \cap T| \leq t - (q+1) < m - (r-1)$. Thus in this case $V_0 \cap T$ could not cover the (at least $m - (r-1)$) edges of \mathcal{H} obtained from \mathcal{L}_1 and uncovered by $V_1 \cap T$.

Form a set T^* by replacing each subset $V_i \cap T$ by an element y_i for $1 \leq i \leq q+1$ whenever this set is non-empty, $V_i \cap T \neq \emptyset$. Then T^* is a cover of size at most $|T|$ of the projective plane corresponding to \mathcal{A}_q . Blokhuis' result (1) implies that T^* contains a line L of the projective plane. Our argument in the previous paragraph gives that L is not the line of infinity, thus it is of the form $L_i^j \cup \{y_i\}$. We may suppose that $i = 1$, i.e., $L_1^j \subset T$ for some $1 \leq j \leq q$. As L_1^j meets all the lines L_i^k of the affine plane for $i > 1$, we may also suppose that $(V_2 \cup \dots \cup V_{q+1}) \cap T = \emptyset$.

In the case of $1 \leq j \leq m$ one needs at least $(q-m)$ vertices (inside $\cup_{k>m} L_1^k$) to cover the hyperedges $\{L_1^k : k > m\}$ and at least $r-2$ more (inside $\cup_{1 \leq k \leq m} L_1^k \cup V_1$) to cover the hyperedges of the form $L_1^k \cup E_1^k$ for $1 \leq k \leq m$, $k \neq j$. In the case of $j > m$ one needs another $(q-m-1)$ vertices of T inside $\cup_{k>m} L_1^k$ and at least $r-1$ more inside $\cup_{1 \leq k \leq m} L_1^k \cup V_1$. In both cases we obtain $|T| \geq t$.

Finally, we show that \mathcal{H} is critical, i.e., removing any edge E one has $\tau(\mathcal{H} \setminus \{E\}) \leq t-1$. Indeed, removing $L_i^j \cup E_i^j$ (for $1 \leq j \leq m$) one can define the set T_i^j as follows. $T_i^j := (V_i \setminus E_i^j) \cup L_i^{m+1} \cup \{x_i^k : k > m+1\}$, where x_i^k is an arbitrary element of L_i^k . This T_i^j has $t-1$ vertices and meets all edges of \mathcal{H} but $L_i^j \cup E_i^j$. Removing L_i^j (for $j > m$) from \mathcal{H} one can construct again a set T_i^j covering all the other edges as follows. $T_i^j := L_i^1 \cup (V_i \setminus E_i^1) \cup \{x_i^k : k > m, k \neq j\}$.

Proof of Theorem 1. It is well-known that primes are relatively dense among integers, e.g., it follows from [11] that for every integer $\tau > \tau_0$ one can find a prime q in the interval $\tau - \tau^{3/5} < q < \tau - 2\sqrt{\tau}$. Then $\ell(\tau) \geq q^2 + q = \tau^2 - O(\tau^{8/5})$ follows from (4), (5) and from the strict monotonicity of ℓ , $\ell(\tau-1) < \ell(\tau)$. (More is true, considering vertex disjoint examples we even get $\ell(a+b) \geq \ell(a) + \ell(b)$.)

Q.E.D.

4 Generalized construction from double critical hypergraphs

In the same way as above one can prove the following statement. Let q be an odd prime, $q \geq 7$, and let \mathcal{A}_q be the Desarguesian affine plane of order q with parallel classes \mathcal{L}_i and vertex set V_0 . Let $\mathcal{S} = \{E^1, \dots, E^m\}$ be a linear, τ -critical hypergraph with covering number $t \geq 1$ such that $m < q$ and $m - t \geq \frac{1}{2}(q - 5)$. Consider $q + 1$ copies of \mathcal{S} with pairwise disjoint vertex sets also disjoint to V_0 . We obtain $\{E_i^1, E_i^2, \dots, E_i^m\}$ for $1 \leq i \leq q + 1$. Define the edge-set of \mathcal{H} in the same way as in (3), then \mathcal{H} is a linear, τ -critical hypergraph with

$$\tau(\mathcal{H}) = q + (t - 1) + (q - m) . \quad (6)$$

The construction can be extended even to the case $m = q$ if the hypergraph \mathcal{S} is *double critical*. We call a τ -critical hypergraph \mathcal{S} double critical if for every $E \in \mathcal{E}(\mathcal{S})$ there exists another edge E' such that removing both of them the covering number decreases by 2, $\tau(\mathcal{S} \setminus \{E, E'\}) = \tau(\mathcal{S}) - 2$. The Desarguesian affine plane \mathcal{A}_q is double critical ($q \geq 3$, a prime number), and any vertex disjoint union of τ -critical hypergraphs yields double critical systems.

Replace the complete graph in the construction of the previous Section by an (almost) optimal double critical linear hypergraph. Induction implies that for $\tau = q + o(\sqrt{q})$ (where q is a prime) one has

$$\ell(\tau) \geq \tau^2 - (2 + o(1))\tau^{3/2}. \quad (7)$$

This is the natural limit of our method in improving the error term for $\ell(\tau)$. However, our constructions are not necessary optimal (in some cases they are not even maximal) so it might be that Sudakov's upper bound is closer to the truth.

5 A remark on set-pair systems

The usual approach for dealing τ -critical hypergraphs is by the so-called set-pair method. Call a collection of pair of sets $(A_i, B_i)_{i=1,2,\dots,m}$ *cross-intersecting* of size m if it consists of disjoint pairs $A_i \cap B_i = \emptyset$ for every i but otherwise $A_i \cap B_j \neq \emptyset$ holds for every $i \neq j$. It is called a cross-intersecting (a, b) -system if in addition $|A_i| \leq a$ and $|B_i| \leq b$ hold for all $1 \leq i \leq m$. The set-pair method was started by Bollobás [6] who proved that the maximum size of a cross-intersecting (a, b) -system, is exactly $\binom{a+b}{a}$. Since then (1965) several generalizations (e.g., Alon [2]) and applications were proved, see the surveys [10] and [17] for more details.

Call a cross-intersecting set-pair system $(A_i, B_i)_{i=1,2,\dots,m}$ an $(a, b)^*$ -system if $|A_i| \leq a$ and $|B_i \cap B_j| < b$ hold for all $1 \leq i \neq j \leq m$. Denote by $f(a, b)$ the maximum possible size of an $(a, b)^*$ -system. From every linear τ -critical hypergraph \mathcal{H} , with $\mathcal{E}(\mathcal{H}) = \{B_1, \dots, B_m\}$ one can construct a $(\tau - 1, 2)^*$ -system of size m in the following natural way. By definition, for any edge B_i there exists a subset A_i of size at most $\tau - 1$ meeting all edges of \mathcal{H} except B_i .

The conjecture in [1] was originally formulated in the stronger form $f(a, b) \leq \binom{a+b}{b}$. The Construction in Section 3 shows that it does not hold for $b = 2$ and $a > a_0$. In general, the best upper bound is due to Sudakov [15], $f(a, b) \leq a^b$. It follows from the recursion $f(a, b) \leq af(a, b-1) + 1$. It might give the right order of magnitude of f .

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