Quadruple Systems with Independent Neighborhoods

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Abstract

A 4-graph is **odd** if its vertex set can be partitioned into two sets so that every edge intersects both parts in an odd number of points. Let

$$b(n) = \max_{\alpha} \left\{ \alpha \binom{n-\alpha}{3} + (n-\alpha)\binom{\alpha}{3} \right\} = \left(\frac{1}{2} + o(1)\right)\binom{n}{4}$$

denote the maximum number of edges in an n-vertex odd 4-graph. Let n be sufficiently large, and let G be an n-vertex 4-graph such that for every triple xyz of vertices, the neighborhood $N(xyz) = \{w : wxyz \in G\}$ is independent. We prove that the number of edges of G is at most $b(n)$. Equality holds only if G is odd with the maximum number of edges. We also prove that there is $\varepsilon > 0$ such that if a 4-graph G has minimum degree at least $(1/2 - \varepsilon)\binom{n}{3}$, then G is 2-colorable.

Our results can be considered as a generalization of Mantel's theorem about triangle-free graphs, and we pose a conjecture about k-graphs for larger k as well.

1 Introduction

Let G be a k-uniform hypergraph (k-graph for short). The neighborhood of a vertex subset $S \subset V(G)$ of size $k-1$ is $N_G(S) = \{v : S \cup \{v\} \in G\}$ (we associate G with its edge set, and will often omit the subscript G). Suppose we impose the restriction that all neighborhoods of G are *independent sets* (that is, span no edges), and G has n vertices. What is the maximum number of edges that G can have? When $k = 2$, the answer is $\lfloor n^2/4 \rfloor$, achieved by the complete bipartite graph $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$. This result, due originally to Mantel in 1907, was the first result of extremal graph theory. Recently, the same question was answered for $k = 3$, where the unique

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extremal example (for n large) is obtained by partitioning the vertex set into two parts X, Y, where $|\,|X| - 2n/3\,| < 1$, and taking all triples with two points in X. This was proved by Füredi, Pikhurko, and Simonovits [3, 4], and settled a conjecture of Mubayi and Rödl [7].

In this paper, we settle the next case, namely $k = 4$. It is noteworthy that determining exact results for extremal problems about k-graphs is in general a hard problem. Consequently, our proof is by no means a straightforward generalization of the corresponding proofs for $k = 2$ and 3, and at present, we do not see how to generalize it to larger k.

Let F^k be the k-graph with $k+1$ edges, k of which share a common vertex set of size $k-1$, and the last edge contains the remaining vertex from each of the first k edges. Writing $[a,b] = \{a, a+1, \dots, b-1, b\}$ (with $[a,b] = \emptyset$ if $a > b$) and $[n] = \{1, \dots, n\}$, a formal description is

$$F^k = \{[k+i] \setminus [k, k+i-1] : 0 \le i \le k-1\} \cup ([2k-1] \setminus [k-1]).$$

Note that a k-graph contains no copy of F^k (as a not necessarily induced subsystem) if and only if each of its neighborhoods is independent.

Call a 4-graph **odd** if its vertex set can be partitioned into $X \cup Y$, such that every edge intersects X in an odd number of points. Let $B(n)$ be one of at most two odd 4-graphs on n vertices with the maximum number of edges and let $b(n) = |B(n)|$. Note that the vertex partition of $B(n)$ is not into precisely equal parts, but they have sizes $n/2 - t$ and $n/2 + t$, where, as it follows from routine calculations,

$$|\,t - \frac{1}{2}\sqrt{3n-4}\,| < 1.$$

It is easy to check that an odd 4-graph has independent neighborhoods, and one might believe that among all n-vertex 4-graphs with independent neighborhoods, the odd ones have the most edges. Our first result confirms this for large n.

Theorem 1.1 (Exact Result) *Let n be sufficiently large, and let G be an n-vertex 4-graph with all neighborhoods being independent sets. Then $|G| \le b(n)$, and if equality holds, then $G = B(n)$.*

We also prove an approximate structure theorem, which states that if G has close to $b(n)$ edges, then the structure of G is close to $B(n)$.

Theorem 1.2 (Global Stability) *For every $\delta > 0$, there exists n_0 such that the following holds for all $n > n_0$. Let G be an n-vertex 4-graph with independent neighborhoods, and $|G| > (1/2 - \varepsilon)\binom{n}{4}$, where $\varepsilon = \delta^2/108$. Then G can be made odd by removing at most $\delta\binom{n}{4}$ edges.*

One might suspect that Theorem 1.2 can be taken further, by showing that if G has minimum degree at least $(1/2 - \gamma)\binom{n}{3}$ for some $\gamma > 0$, then G is already odd. Such phenomena hold for $k = 2$ and 3. For example, when $k = 2$, a special case of the theorem of Andrásfai, Erdős, and

Sós [1] states that a triangle-free graph with minimum degree greater than $2n/5$ is bipartite. For $k=3$, a similar result was proved in [4]. The analogous statement is not true for $k=4$. Indeed, one can add an edge E to $B(n)$ that intersects each part in two vertices, and then delete all edges of $B(n)$ that intersect E in three vertices. The resulting 4-graph has independent neighborhoods, and yet its minimum degree is $(1/2)\binom{n}{3} - O(n^{5/2})$. Nevertheless, a slightly weaker statement is true. Let us call a k-graph 2-*colorable* if its vertex set can be partitioned into two independent sets.

Theorem 1.3 *Let G be an n-vertex 4-graph with independent neighborhoods. There exists $\varepsilon > 0$ such that if n is sufficiently large and G has minimum degree greater than $(1/2-\varepsilon)\binom{n}{3}$, then G is 2-colorable.*

Call a k-graph **odd** if it has a vertex partition $X \cup Y$, and all edges intersect X in an odd number of points less than k. Let $B^k(n)$ be an odd k-graph with the maximum number of edges (this may not be unique).

Conjecture 1.4 *Let n be sufficiently large and let G be an n-vertex k-graph with independent neighborhoods. Then $|G| \le |B^k(n)|$, and if equality holds, then $G = B^k(n)$.*

2 Asymptotic Result and Stability

In this section we prove Theorem 1.2. Before doing so we first prove an asymptotic result and a stability result under the assumption of large minimum degree.

Let $\mathrm{ex}(n, F^4)$ denote the maximum number of edges in an n-vertex 4-graph containing no copy of F^4. The results of Katona, Nemetz, and Simonovits [5] imply that $\lim_{n\to\infty} \mathrm{ex}(n, F^4)/\binom{n}{4}$ exists. Let the *Turán density* $\pi(F^4)$ be the value of the limit. We need the following standard lemma.

Lemma 2.1 (See Frankl and Füredi [2]) *Let F be a k-graph with the property that every pair of its vertices lies in an edge. Then*

$$\pi(F)\binom{n}{k} \le \mathrm{ex}(n, F) \le \pi(F)\frac{n^k}{k!}.$$

Observe that F^4 satisfies the hypothesis of Lemma 2.1. Write $d_{\min}(G)$ for the minimum vertex degree in G

Theorem 2.2 (Asymptotic Result and Minimum Degree Stability) *For every $\delta > 0$, there exists n_1 such that the following holds for all $n > n_1$. Let G be an n-vertex 4-graph with independent neighborhoods and $d_{\min}(G) > (\pi(F^4) - \delta/24)\binom{n}{3}$. Then G can be made odd by deleting at most $\delta\binom{n}{4}$ edges. Also, $\pi(F^4) = 1/2$.*

Proof. Suppose $\delta > 0$ is given, and set $\gamma = \delta/24 < 1/24$. Let $\pi = \pi(F^4)$. Note that $B(n)$ shows that $\pi \geq 1/2$. Let A be a maximum size neighborhood in G. By hypothesis, A is an independent set. Put $B = V \setminus A$, and $\mu = |A|$. Since $d_{\min}(G) > (\pi - \gamma)\binom{n}{3}$, we have $|G| > (\pi - \gamma)\binom{n}{3}(n/4)$, and therefore $\mu > (\pi - \gamma)n$. Let H_i be the set of edges in G with precisely i vertices in B, and $h_i = |H_i|$. Observe that $h_0 = 0$ since A is an independent set. Recalling that $|G| \leq \pi n^4/24$ by Lemma 2.1, we have

$$\sum_{i=1}^{4} i \cdot h_i = \sum_{x \in B} \deg(x) = 4|G| - \sum_{x \in A} \deg(x) < 3|G| + \pi \frac{n^4}{24} - \mu(\pi - \gamma)\binom{n}{3}. \tag{1}$$

Let $\sum_{AAB}$ denote the summation of $|N_G(S)|$ over all sets $S = \{u, v, w\}$, with $u, v \in A$ and $w \in B$. By the definition of A, each of these terms is at most μ. Consequently,

$$3h_1 + 2h_2 = \sum_{AAB} \leq \mu(n - \mu)\binom{\mu}{2}. \tag{2}$$

Now we add (1) and 2/3 times (2). Using $|G| = \sum_{i=1}^{4} h_i$, we obtain

$$\frac{h_2}{3} + h_4 < \gamma\mu\frac{n^3}{6} + \frac{1}{3}\mu^3(n - \mu) + \frac{\pi}{24}(n - 4\mu)n^3 + O(n^2).$$

The right hand side simplifies to

$$\gamma\mu\frac{n^3}{6} + \frac{1}{48}(2\mu + n)(n - 2\mu)^3 + \frac{\pi - 1/2}{24}(n - 4\mu)n^3 + O(n^2).$$

Since $2n > 2\mu > 2(\pi - \gamma)n \geq (1 - 2\gamma)n$, the second summand above is at most $(\gamma^3/2)n^4$. If $\pi \geq 1/2 + 3\gamma$, then $\mu > n/2$ and

$$\gamma\mu\frac{n^3}{6} + \frac{\pi - 1/2}{24}(n - 4\mu)n^3 \leq -\frac{\gamma}{24}n^4.$$

This implies that $h_2/3 + h_4$ is negative, which is a contradiction. Consequently, $\pi < 1/2 + 3\gamma$, and since γ can be arbitrarily close to 0, we conclude that $\pi = 1/2$.

Using $\pi = 1/2$ and $n > n_1$ now yields $h_2/3 + h_4 < (\gamma/6 + \gamma^3/2)n^4 < 8\gamma\binom{n}{4}$. Therefore $h_2 + h_4 < 24\gamma\binom{n}{4} = \delta\binom{n}{4}$. Since we have already argued that $h_0 = 0$, the vertex partition A, B satisfies the requirements of the theorem, and the proof is complete. □

Proof of Theorem 1.2. The proof is a standard reduction to Theorem 2.2. Let $\delta > 0$ be given. We can assume that $\delta < 1$. Suppose that n_1 is the output of Theorem 2.2 with input $\delta/2$. Set $\gamma = \delta/48$, and let $n > n_1/(1 - \delta)$ be sufficiently large. Let $G_n = G$ be the given 4-graph G with the properties in Theorem 1.2.

If the current 4-graph G_i with i vertices has a vertex x of degree at most $(1/2 - \gamma)\binom{i}{3}$, then remove x obtaining the new 4-graph G_{i-1}, and repeat; otherwise we terminate the procedure. Let G_m be the final graph. By Lemma 2.1,

$$\begin{aligned}\frac{m^4}{48} \;\geq\; |G_m| \;\geq\; & \left(\frac{1}{2}-\varepsilon\right)\binom{n}{4} - \left(\frac{1}{2}-\gamma\right)\sum_{i=m+1}^{n}\binom{i}{3} \\ =\; & (\gamma-\varepsilon)\frac{n^4}{24} + (1-2\gamma)\frac{m^4}{48} + O(n^3).\end{aligned}$$

It follows that

$$m/n \geq (1-\varepsilon/\gamma)^{1/4} + o(1) > 1 - \varepsilon/4\gamma = 1 - \delta/9$$

and $m > n_1$. Applying Theorem 2.2 to the 4-graph G_m of minimum degree at least $(1/2-\gamma)\binom{m}{3}$, we obtain a partition $X \cup Y$ of $V(G_1)$ with all but $(\delta/2)\binom{m}{4}$ edges having even intersection with the parts. We removed at most $\delta n/9$ vertices (and thus at most $(\delta/2)\binom{n}{4}$ edges) from G to form G_m. Therefore, we can remove at most $\delta\binom{n}{4}$ edges from G to make it odd. □

3 A Magnification Lemma

Given a vertex partition of $V(G)$, call an edge *odd* if it intersects either part in an odd number of vertices, and *even* otherwise. Let $\mathcal{M}$ denote the set of quadruples intersecting either part in an odd number of points that are *not* in G. Let $\mathcal{B}$ denote the set of even edges in G. Call a partition $V(G) = X \cup Y$ a *maximum cut* of G if it minimizes $|\mathcal{B}|$. Sometimes we denote a typical edge $\{w,x,y,z\}$ simply by $wxyz$. Let $a \pm b$ denote the interval $(a-b, a+b)$ of reals.

Lemma 3.1 *Let n be sufficiently large and let G be an n-vertex 4-graph with independent neighborhoods and $d_{\min}(G) \geq (1/2 - 10^{-40})\binom{n}{3}$. Let X, Y be a maximum cut of G, and suppose that $|X|$ and $|Y|$ are both in $(1/2 \pm 10^{-15})n$. If $|\mathcal{M}| \leq n^4/10^{40}$, then every vertex w of G satisfies $\deg_{\mathcal{B}}(w) \leq n^3/10^9$.*

Proof. Suppose, for a contradiction, that there is a vertex $w \in X$ with $\deg_{\mathcal{B}}(w) > n^3/10^9$. Say that an edge is of the form X^iY^j if it has i points in X and j points in Y (for $i+j=4$). We partition the argument into two cases.

Case 1. At least $n^3/(2\cdot 10^9)$ edges of $\mathcal{B}$ containing w are of the form $XXXX$.

Now w is in at least as many odd edges as even edges, else we could move w from X to Y. So in particular, since $\deg_G(w) \geq d_{\min}(G) > 2\binom{n}{3}/5$, we conclude that w is in at least $\binom{n}{3}/5$ odd edges. At least $\binom{n}{3}/10$ of these are $XYYY$ edges or at least $\binom{n}{3}/10$ of these are $XXXY$ edges. Depending on which choice occurs, call the resulting set of edges $\mathcal{H}$.

For every choice of $x, y, z \in X$, with $E = \{w,x,y,z\} \in \mathcal{B} \subset G$, and for every choice of $E' = \{v_1, v_2, v_3, w\} \in \mathcal{H} \subset G$ with $E \cap E' = \{w\}$, consider the five quadruples

$$v_1v_2v_3w,\ v_1v_2v_3x,\ v_1v_2v_3y,\ v_1v_2v_3z,\ wxyz.$$

Regardless of whether E' is of the form $XYYY$ or $XXXY$, the first four quadruples are odd. The first and fifth quadruple are both in G, so one of the middle three must be in $\mathcal{M}$. On the other hand, each such quadruple D is counted at most $3n^2$ times (note that w is fixed, so in the case of $XYYY$ edges we only have to choose the remaining two points in E; in the case of $XXXY$ edges, we also may choose the unique point of $E \cap D$ thereby giving the additional factor of 3). Putting this together we have

$$|\mathcal{M}| \geq \frac{n^3}{2 \cdot 10^9} \times \frac{\binom{n}{3}/10 - 2n^2}{3n^2} > \frac{n^4}{10^{40}}$$

which is a contradiction.

Case 2. At least $n^3/(2 \cdot 10^9)$ edges of $\mathcal{B}$ containing w are of the form $XXYY$.

First suppose that at least $\binom{n}{3}/10^{20}$ odd edges containing w are of the form $XYYY$. For every choice of $x \in X$, $y, z \in Y$, with $E = \{w, x, y, z\} \in \mathcal{B} \subset G$, and for every choice of an odd edge $E' = \{v_1, v_2, v_3, w\} \in G$ with $E \cap E' = \{w\}$, consider the five quadruples

$$xyzw, xyzv_1, xyzv_2, xyzv_3, wv_1v_2v_3.$$

One of the three middle quadruples must be in $\mathcal{M}$ and each such quadruple is counted at most $3n^2$ times (note that w is fixed, so we only have to choose the remaining two points in E' and the two points of $E \cap \{y, z, v_i\}$). Putting this together we have

$$|\mathcal{M}| \geq \frac{n^3}{2 \cdot 10^9} \times \frac{\binom{n}{3}/10^{20} - 2n^2}{3n^2} > \frac{n^4}{10^{40}}$$

which is a contradiction. Consequently, we may assume that

(i) the number of $XYYY$ edges containing w is at most $\binom{n}{3}/10^{20}$, and

(ii) the number of $XXXX$ edges containing w is at most $n^3/(2 \cdot 10^9)$ (otherwise we use Case 1).

Statements (i) and (ii) imply that the edges of G containing w are essentially of two types: $XXXY$, and $XXYY$. Define the 3-graph $L(w) = \{\{a, b, c\} : \{w, a, b, c\} \in G\}$. By hypothesis

$$|L(w)| = \deg_G(w) \geq \left(\frac{1}{2} - \frac{1}{10^{40}}\right)\binom{n}{3}.$$

Partition $L(w)$ as

$$L_{XXX} \cup L_{XXY} \cup L_{XYY} \cup L_{YYY},$$

where $L_{X^iY^j}$ is the set of edges of L with i points in X and j points in Y $(i + j = 3)$. Again, (i) and (ii) imply that $|L_{XXX}| + |L_{YYY}| < \binom{n}{3}/10^5$, so

$$|L_{XXY}| + |L_{XYY}| > \left(\frac{1}{2} - \frac{1}{10^4}\right)\binom{n}{3}.$$

For every pair $a \in X, b \in Y$, let $d(a, b)$ denote the number of triples $\{a, b, c\} \in L(w)$. Then

$$\sum_{a \in X, b \in Y} d(a, b) = 2(|L_{XXY}| + |L_{XYY}|) > \left(1 - \frac{2}{10^4}\right)\binom{n}{3}.$$

Consequently, recalling that $|X|$ and $|Y|$ are both in $(1/2 \pm 10^{-15})n$, there exist $a_0 \in X$ and $b_0 \in Y$, for which

$$d(a_0, b_0) > \frac{1 - 2 \cdot 10^{-4}}{|X||Y|} \binom{n}{3} > \frac{1 - 2 \cdot 10^{-4}}{(1/4 + 2 \cdot 10^{-15})n^2} \binom{n}{3} > \left(\frac{2}{3} - \frac{1}{10^3} \right) n.$$

We conclude that there exist $S \subset X$ and $T \subset Y$, each of size at least $(2/3 - 1/2 - 10^{-2})n = (1/6 - 10^{-2})n$ such that $\{w, a_0, b_0, s\}, \{w, a_0, b_0, t\} \in G$ for every $s \in S$ and $t \in T$.

For every choice of distinct $s, s', s'' \in S$, and $t \in T$, consider the five quadruples

$$wa_0b_0s, wa_0b_0s', wa_0b_0s'', wa_0b_0t, ss's''t.$$

Since the first four are in G, we must have $\{s, s', s'', t\} \in \mathcal{M}$. Consequently,

$$|\mathcal{M}| \geq \binom{|S|}{3} |T| > \binom{(1/6 - 10^{-2})n}{3} (1/6 - 10^{-2})n > \frac{n^4}{10^{40}}.$$

This contradiction completes the proof of the lemma. □

4 The Exact Result

Proof of Theorem 1.1. Let G be an n-vertex 4-graph with independent neighborhoods and $|G| = b(n)$. Since $B(n)$ is maximal with respect to the property of being F^4-free, it suffices to show that $G = B(n)$.

We claim that we may also assume that $d_{\min}(G) \geq b(n) - b(n-1)$. Indeed, otherwise, assuming we have proved the result under this assumption for $n > n_0$, we can successively remove vertices of small degree to obtain a contradiction. (Note that each removal strictly increases the difference $|G| - b(n)$, where n is the number of vertices in G.) We refer the Reader to Keevash and Sudakov [6, Theorem 2.2] for the details. Also in [6] we have the calculations showing that

$$d_{\min}(G) \geq b(n) - b(n-1) > \frac{1}{12} n^3 - \frac{1}{2} n^2 > \left(\frac{1}{2} - \frac{1}{10^{40}} \right) \binom{n}{3}.$$

Choose a maximum cut $X \cup Y$ of G. By Theorem 1.2, we may assume that the number of even edges is less than $n^4/10^{40}$ (choose n sufficiently large to guarantee this). It also follows that, for example, $|X|$ and $|Y|$ both lie in $(1/2 \pm 10^{-15})n$ for otherwise a short calculation shows that $|G| < b(n)$. These bounds will be used throughout.

Define $\mathcal{M}$ and $\mathcal{B}$ as in Section 3. Call quadruples in $\mathcal{M}$ *missing* and those in $\mathcal{B}$ *bad*. Since $(G \cup \mathcal{M}) \setminus \mathcal{B}$ is odd and $|G| = |B(n)|$, we conclude that

$$|B(n)| + |\mathcal{M}| - |\mathcal{B}| = |G| + |\mathcal{M}| - |\mathcal{B}| \leq |B(n)| \tag{3}$$

and therefore $|\mathcal{B}| \geq |\mathcal{M}|$. In particular, this implies that $|\mathcal{M}| < n^4/10^{40}$. If $\mathcal{B} = \emptyset$, then G is odd, so $G = B(n)$ and we are done. Hence assume that $\mathcal{B} \neq \emptyset$. In the remainder of the proof, we will obtain a contradiction to $|\mathcal{M}| < n^4/10^{40}$, or to the choice of the partition of $V(G)$.

Our strategy is to show that each even edge yields many potential copies of F^4, and hence many missing quadruples. Define

$$A = \{z \in V(G) : \deg_{\mathcal{M}}(z) > n^3/10^7\}.$$

Our first goal is to prove that $A \neq \emptyset$. In fact, we actually will need the following stronger statement:

Claim. There exists $\mathcal{B}' \subset \mathcal{B}$ such that $|\mathcal{B}'| > |\mathcal{B}|/20$ and

$$\forall E \in \mathcal{B}', \qquad |E \cap A| \geq 1. \tag{4}$$

Proof of Claim. Write $\mathcal{B} = \mathcal{B}_{XXXX} \cup \mathcal{B}_{YYYY} \cup \mathcal{B}_{XXYY}$ (with the obvious meaning).

Case 1. $|\mathcal{B}_{XXXX}| + |\mathcal{B}_{YYYY}| \geq |\mathcal{B}|/10$.

Pick $E = \{w, x, y, z\} \in \mathcal{B}_{XXXX} \cup \mathcal{B}_{YYYY}$. Assume without loss of generality that $\{w, x, y, z\} \in \mathcal{B}_{XXXX}$. For every choice of $v_1, v_2, v_3 \in Y$ the five quadruples

$$v_1v_2v_3w, v_1v_2v_3x, v_1v_2v_3y, v_1v_2v_3z, wxyz \tag{5}$$

form a potential copy of F^4, so one of the first four must be in $\mathcal{M}$. This gives $|\mathcal{M}| \geq \binom{|Y|}{3}$, and so at least $\binom{|Y|}{3}/4 > n^3/10^7$ of these quadruples of $\mathcal{M}$ contain the same vertex of E, say w. Thus $deg_{\mathcal{M}}(w) > n^3/10^7$. Now let $\mathcal{B}' = \mathcal{B}_{XXXX} \cup \mathcal{B}_{YYYY}$. Then $|\mathcal{B}'| \geq |\mathcal{B}|/10 > |\mathcal{B}|/20$ as claimed.

Case 2. $|\mathcal{B}_{XXYY}| > 9|\mathcal{B}|/10$.

Let $\mathcal{B}' = \{E \in \mathcal{B} : |E \cap A| \geq 1\}$. If $|\mathcal{B}'| \geq |\mathcal{B}_{XXYY}|/10$, then

$$|\mathcal{B}'| \geq \frac{|\mathcal{B}_{XXYY}|}{10} > \frac{1}{10} \times \frac{9}{10}|\mathcal{B}| > \frac{|\mathcal{B}|}{20}$$

and we are done. Hence we may assume that $|\mathcal{B}'| < |\mathcal{B}_{XXYY}|/10$. Let $\mathcal{B}'' = \mathcal{B}_{XXYY} \setminus \mathcal{B}'$. Thus $|\mathcal{B}''| > 9|\mathcal{B}_{XXYY}|/10$. Given a set S of vertices, write $\deg_{\mathcal{M}}(S)$ for the number of edges of $\mathcal{M}$ containing S.

Subclaim. For every $E \in \mathcal{B}''$, and for every $S \in \binom{E}{3}$, we have $\deg_{\mathcal{M}}(S) \geq (1/2 - 10^{-2})n$.

Proof of Subclaim. Suppose to the contrary that there exists $E \in \mathcal{B}''$ and $S \in \binom{E}{3}$ with $deg_{\mathcal{M}}(S) < (1/2 - 10^{-2})n$. Assume that $E = \{w, x, y, z\}$ with $w, x \in X$ and $y, z \in Y$ and $S = \{x, y, z\}$. Let $Y' = \{v \in Y : \{x, y, z, v\} \in G\}$. Then

$$|Y'| \geq |Y| - \deg_{\mathcal{M}}(S) - 2 > \left(\frac{1}{2} - \frac{1}{10^{14}} - \frac{1}{2} + \frac{1}{10^2}\right) n = \left(\frac{1}{10^2} - \frac{1}{10^{14}}\right) n.$$

For every choice of $v_1, v_2, v_3 \in Y'$ the five quadruples

$$xyzv_1, xyzv_2, xyzv_3, xyzw, v_1v_2v_3w.$$

form a potential copy of F^4, so the last one must be in $\mathcal{M}$. This gives

$$\deg_{\mathcal{M}}(w) > \binom{|Y'|}{3} \geq \binom{(10^{-2} - 10^{-14})n}{3} > \frac{n^3}{10^7}.$$

Consequently, $E \in \mathcal{B}'$ which contradicts the fact that $\mathcal{B}' \cap \mathcal{B}'' = \emptyset$. $\square$

Counting edges of $\mathcal{M}$ from subsets of edges of $\mathcal{B}''$ yields

$$\binom{3}{2} \cdot \max\{|X|, |Y|\} \cdot |\mathcal{M}| \geq \sum_{E \in \mathcal{B}''} \sum_{S \in \binom{E}{3}} deg_{\mathcal{M}}(S),$$

since the right hand side counts an edge of $\mathcal{M}$ at most $3\max\{|X|, |Y|\}$ times. For example, an edge $\{a, b, c, d\} \in \mathcal{M}$ with $a \in X$ and $b, c, d \in Y$ is counted on the right-hand side by choosing $E \in \mathcal{B}''$ where $|E \cap \{b, c, d\}| = 2$ and $a \in E$. Using $|\mathcal{B}''| \geq (0.9)|\mathcal{B}_{XXYY}| > (0.9)^2|\mathcal{B}| \geq (0.9)^2|\mathcal{M}|$, and the Subclaim, we get

$$|\mathcal{M}| \geq \frac{(0.9)^2 \cdot 4(1/2 - 10^{-2})n}{3 \cdot (1/2 + 10^{-15})n} |\mathcal{M}| = 1.08 \left(\frac{1/2 - 10^{-2}}{1/2 + 10^{-15}} \right) |\mathcal{M}| > |\mathcal{M}|.$$

This contradiction concludes the proof of Case 2 and of the Claim. $\square$

Counting missing edges from vertices of A, we have

$$4|\mathcal{M}| \geq \sum_{x \in A} \deg_{\mathcal{M}}(x) > \frac{|A|n^3}{10^7}.$$

Recalling that $|\mathcal{B}'| > |\mathcal{B}|/20$ and $|\mathcal{B}| \geq |\mathcal{M}|$, we obtain

$$|\mathcal{B}'| > \frac{|\mathcal{M}|}{20} > \frac{|A|}{80} \frac{n^3}{10^7}.$$

Now the Claim (see (4)) implies that

$$\sum_{x \in A} \deg_{\mathcal{B}'}(x) \geq |\mathcal{B}'| > \frac{|A|}{80} \frac{n^3}{10^7}.$$

Consequently, there exists $w \in V(G)$ for which $\deg_{\mathcal{B}}(w) \geq \deg_{\mathcal{B}'}(w) > n^3/(80 \cdot 10^7) > n^3/10^9$. This contradicts Lemma 3.1 and completes the proof of the theorem. $\square$

5 The Sharp Structure

Proof of Theorem 1.3. Let $\delta = 12/10^{40}$, and choose $\varepsilon < \delta/12$ from Theorem 1.2. Now $|G| > (1/2 - \varepsilon)\binom{n}{4}$, so by Theorem 1.2 G has a vertex partition $X \cup Y$ with the number of even edges less than $\delta\binom{n}{4} < n^4/(2 \cdot 10^{40})$. Easy calculations show that $|X|$ and $|Y|$ are both in $(1/2 \pm 10^{-15})n$. We may also assume that X, Y is a maximum cut. We will show that both X and Y are independent sets. As in (3), we have

$$\left(\frac{1}{2} - \varepsilon\right)\binom{n}{4} + |\mathcal{M}| - |\mathcal{B}| < |G| + |\mathcal{M}| - |\mathcal{B}| \leq b(n)$$

which implies that

$$|\mathcal{M}| \leq |\mathcal{B}| + b(n) - \left(\frac{1}{2} - \varepsilon\right)\binom{n}{4} \leq \frac{n^4}{2 \cdot 10^{40}} + \varepsilon\binom{n}{4} + O(n^3) < \frac{n^4}{10^{40}}.$$

Suppose now that there is an edge E of G in $\binom{X}{4} \cup \binom{Y}{4}$. Assume by symmetry that $E \in \binom{X}{4}$. Then by the same argument as in (5), we obtain $\deg_{\mathcal{M}}(w) > \binom{|Y|}{3}/4 > n^3/10^5$ for some $w \in E$. Now

$$\left(\frac{1}{2} - \varepsilon\right)\binom{n}{3} < \deg_G(w) = \deg_{\mathcal{B}}(w) + \left(\binom{|Y|}{3} + \binom{|X|-1}{2}|Y| - \deg_{\mathcal{M}}(w)\right).$$

As $\binom{|Y|}{3} + \binom{|X|-1}{2}|Y| < (1/2+\varepsilon)\binom{n}{3}$ we obtain $\deg_{\mathcal{B}}(w) \geq n^3/10^5 - 2\varepsilon\binom{n}{3} > n^3/10^9$. This contradicts Lemma 3.1 and completes the proof. □

References

[1] B. Andrásfai, P. Erdős, V.T. Sós, On the connection between chromatic number, maximal clique, and minimal degree in a graph, Discrete Math, 8 (1974) 205–218.

[2] P. Frankl, Z. Füredi, Extremal problems and the Lagrange function of hypergraphs, Bulletin of the Institute of Mathematics, Academia Sinica, 16, (1988) 305–313.

[3] Z. Füredi, O. Pikhurko, and M. Simonovits, The Turán density of the hypergraph {*abc*, *ade*, *bde*, *cde*}, Electronic J. Combinatorics 10 (2003), 8 pp.

[4] Z. Füredi, O. Pikhurko, and M. Simonovits, On triple systems with independent neighborhoods, Combinatorics, Probability & Computing, 14 (2005) 795–813.

[5] G. O. H. Katona, T. Nemetz, and M. Simonovits, On a graph problem of Turán (In Hungarian), Mat. Fiz. Lapok 15 (1964), 228–238.

[6] P. Keevash, B. Sudakov, On a hypergraph Turán problem of Frankl, Combinatorica, 25 (2005) 673–706.

[7] D. Mubayi, V. Rödl, On the Turán number of triple systems, Journal of Combinatorial Theory, Series A, 100 (2002) 135–152.