

Note

On r -Cover-free Families

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A very short proof is presented for the almost best upper bound for the size of an r -cover-free family over n elements. © 1996 Academic Press, Inc.

A family of sets \mathcal{F} is called r -cover-free if $A_0 \not\subseteq A_1 \cup A_2 \cup \dots \cup A_r$ holds for all distinct $A_0, A_1, \dots, A_r \in \mathcal{F}$. Let $T(n, r)$ denote the maximum cardinality of such an \mathcal{F} over an n -element underlying set. This notion was introduced by Kautz and Singleton [4] in 1964 concerning binary codes. They proved $\Omega(1/r^2) \leq \log T(n, r)/n \leq O(1/r)$ (log is always of base 2). This result was rediscovered several times in information theory, in combinatorics [2], and in group testing [3]. A recent account and related problems can be found in Körner [5]. Dyachkov and Rykov [1] obtained with a rather involved proof that $\log T(n, r)/n \leq O(\log r/r^2)$. Recently, Ruszinkó [6] gave a purely combinatorial proof. Our aim is to present an even simpler argument to show that

$$\frac{\log T(n, r)}{n} \leq \frac{4 \log r + O(1)}{r^2}. \quad (1)$$

This upper bound is twice as good as that of [6], but about half as good as that obtained from the inductive proof of [1]. Our argument is implicitly contained in Erdős, Frankl, and Füredi [2].

THEOREM. *If \mathcal{F} is a family of subsets of an n -element underlying set V such that no set $F_0 \in \mathcal{F}$ is contained in the union of r other members of \mathcal{F} , then*

$$|\mathcal{F}| \leq r + \binom{n}{(n-r)/\binom{r+1}{2}}. \quad (2)$$

Proof. Fix an integer t with $n/2 \geq t > 0$. Define $\mathcal{F}_t \subset \mathcal{F}$ as the family of members having its own t -subset, i.e., $\mathcal{F}_t := \{F \in \mathcal{F} : \text{there exists a } t\text{-element set } A \subseteq F \text{ such that } A \not\subseteq F' \text{ holds for every other } F' \in \mathcal{F}\}$, and let \mathcal{A} be the family of these t -subsets. Let $\mathcal{F}_0 := \{F \in \mathcal{F} : |F| < t\}$, and let \mathcal{B} be the family of t -sets containing a member of \mathcal{F}_0 , i.e., $\mathcal{B} := \{T : T \subset V, |T| = t, \text{ and there exists some } F \in \mathcal{F}_0 \text{ with } T \supset F\}$. The set-system \mathcal{F} is an *antichain*; no two members contain each other. This implies that \mathcal{A} and \mathcal{B} are disjoint. A lemma of Sperner [7] states that $|\mathcal{F}_0| \leq |\mathcal{B}|$. We obtain that $|\mathcal{F}_0 \cup \mathcal{F}_t| \leq |\mathcal{A}| + |\mathcal{B}| \leq \binom{n}{t}$.

Let $\mathcal{F}' := \mathcal{F} \setminus (\mathcal{F}_0 \cup \mathcal{F}_t)$. We claim that $F \in \mathcal{F}', F_1, F_2, \dots, F_i \in \mathcal{F}$ imply

$$\left| F \setminus \bigcup_{j \leq i} F_j \right| > t(r - i). \tag{3}$$

Indeed, if $F \setminus (F_1 \cup \dots \cup F_i)$ can be written as the union of the t -element sets $A_{i+1}, A_{i+2}, \dots, A_r$, then by the choice of F there are members $F \neq F_j \in \mathcal{F}$ with $A_j \subseteq F_j$. Therefore $F \subset (F_1 \cup \dots \cup F_r)$, a contradiction.

Inequality (3) implies that for $F_0, F_1, \dots, F_r \in \mathcal{F}'$ one has $|\bigcup_{i \leq r} F_i| = |F_0| + |F_1 \setminus F_0| + |F_2 \setminus (F_1 \cup F_0)| + \dots + |F_r \setminus (F_0 \cup F_1 \cup \dots \cup F_{r-1})| \geq r + 1 + t \binom{r+1}{2}$. Here the right-hand side exceeds n for $t := \lceil (n-r) / \binom{r+1}{2} \rceil$, implying $|\mathcal{F}'| \leq r$. ■

Finally, the upper bound (1) easily follows from (2) using $\binom{n}{t} \leq n^t/t! < (en/t)^t$.

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