Math 589 Homework

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1. Let $\mathbf{v}_1, \dots, \mathbf{v}_n \in \mathbb{R}^2$, $\|\mathbf{v}_i\| \leq 1$ for all i, and $\sum \mathbf{v}_i = \mathbf{0}$. Prove that there exists an $r \in \mathbb{R}$ (independent of n) and some **permutation** $\pi : [n] \to [n]$ such that, for all $1 \leq k \leq n$, $\|v_{\pi(1)} + v_{\pi(2)} + \cdots + v_{\pi(k)}\| < r$.

2. Prove statement analogous to 1 for vectors in $\mathbb{R}^d$.

3. Let $\mathbf{v}_1, \dots, \mathbf{v}_n \dots \in \mathbb{R}^d$ an infinite sequence of vectors, such that $\sum_i v_i$ is convergent. Let A be the set of all **limit points** after rearrangements, i.e.,

$$A := \{\lim_k (\sum_{1 \leq i \leq k} v_{\pi(i)}) : \pi : Z^+ \to Z^+ \text{a permutation}\}.$$

Prove that A is an affine subspace.
In $d = 1$ (Reiman's thm) this means that either $\limsup |v_i| > 0$ (then $A = \emptyset$), or the sequence is absolute convergent (A is a single point), or every real number can be a limit $A = R$.

4. For $k \in \mathbb{N}$, define $\mathbf{p}_t = (t, t^2, t^3, t^4, t^5) \in \mathbb{R}^5$. Let $\Pi = \text{conv}\{\mathbf{p}_1, \dots, \mathbf{p}_n\}$. Finish the proof that for all triples $i, j, k \in [n]$, $\text{conv}\{\mathbf{p}_i, \mathbf{p}_j, \mathbf{p}_k\}$ is on the **boundary** of Π.

5. Let $P \subset R^4$ =conv$\{\pm \mathbf{e}_i \pm \mathbf{e}_j : i \neq j, i, j = 1, 2, 3, 4\}$, where $\mathbf{e}_1, \dots,$ is the standard basis. (This P is called the **24-cell**.) Describe the face lattice of P.
Describe its **dual** $P^* := \{y \in R^4 : x^T y \leq 1 \text{ for all } x \in P\}$.

6. Show that for any $X \subseteq R^d$, $(X^*)^*$ equals to the closure of conv$(X \cup \{\mathbf{0}\})$, where X^* stands for the dual set to X.

7. Let $\mathcal{C}$ be a family of convex sets in $\mathbb{R}^d$ and suppose that either $|\mathcal{C}| < \infty$ or that all $C \in \mathcal{C}$ are compact. Let $D \subseteq \mathbb{R}^d$. Suppose that for all $(d+1)$-tuples $C_1, \dots, C_{d+1} \in \mathcal{C}$, $C_1 \cap \cdots \cap C_{d+1}$ contains a translated copy of D. Prove that $\cap \mathcal{C}$ contains a **translated copy** of D.

8. Let $X \subset R^d$. Prove that for the **diameters** diam(X)=diam(conv(X)).

9. Let C be a convex set, cl(C) its closure. Prove that diam(C) =diam(cl(C)).

10. Suppose that $P \subseteq \mathbb{R}^d$ and $\mathbf{x} \in \text{int}(\text{conv}(P))$. Prove that for some $A \subseteq P$ with $|A| \leq 2d$, $\mathbf{x} \in \text{int}(\text{conv}(A))$.

11. Prove that there exists a constant $c(d) > 0$ such that the following holds.
Let $\mathcal{C}$ be a family of convex sets in $\mathbb{R}^d$ and suppose that either $|\mathcal{C}| < \infty$ or that all $C \in \mathcal{C}$ are compact. Suppose that for all $2d$-tuples $C_1, \ldots, C_{2d} \in \mathcal{C}$, for the **volume of the intersection** $\mathrm{Vol}(C_1 \cap \cdots \cap C_{2d}) \geq 1$. Then $\mathrm{Vol}(\cap \mathcal{C}) \geq c(d)$.
Show that it does not hold for $(2d-1)$-wise intersections.
Give good estimate for $c(2)$, for $c(d)$.

12. Is the **Lovász bound** $\alpha(G) \leq \vartheta(G)$ always better than the bound $\alpha(G) \leq \alpha^*(G)$?

13. Determine the Shannon capacity of the **Petersen graph**, P_{10}.

14. Show that the center of gravity of a simplex in R^d with vertices $\mathbf{v}_1, \ldots, \mathbf{v}_{d+1}$ is the same as the center of gravity of its vertices.

15. Let $\mathbf{c}$ be the **center of gravity** of a convex and compact set $C \subseteq \mathbb{R}^d$ with nonzero interior. Prove that there is an ellipsoid E such that $\mathbf{c} + E \subseteq C \subseteq \mathbf{c} + dE$.

16. Suppose now that C is **symmetric**, $C = -C$, too. Prove that there exists an ellipsoid E with $E \subseteq C \subseteq \sqrt{d}E$.

17. Suppose that there exists a $\delta(d) > 0$ such that the following holds:
If C is a convex body, and $r(C)$ is the radius of the smallest cylinder containing it, and $\rho(C)$ is the radius of smallest **hole** on a hyperplne such that C can be pushed through, then $\rho \geq \delta r$.

18. Let $\mathcal{C} := \{C_1, \ldots, C_n\}$ be compact convex domains on the plane R^2. Their intersection graph $G := G(\mathcal{C})$ has n vertices $V = \{1, 2, \ldots, n\}$ with ij is an edge if $C_i \cap C_j \neq \emptyset$. Construct a graph G which cannot be represented as an **intersection graph**.

19. Consctruct a graph G which is not an intersection graph of $C_1, \ldots, C_n$ where now every C_i is a **connected**, compact set with C_i =cl(int(C)).

20. It is easy to see that if $f : R^2 \to R$ is a bounded, convex function (i.e., say, $f(\mathbf{x}) \leq 1$), then it is constant. Prove this for **discrete convex functions**:
Suppose $f(x, y) : Z^2 \to R$ is a function on the lattice points, such that $f \leq 1$ and

$$f(x, y) \leq \frac{1}{4}\left(f(x+1, y) + f(x, y+1) + f(x-1, y) + f(x, y-1)\right)$$

holds for every $(x, y) \in Z^2$. Prove that f is constant.

21. A simplicial **decomposition of the the unit cube** $Q := [0, 1]^d$ is a set of simplices $T_1, \ldots, T_m$ such that $\mathrm{int}(T_i) \cap \mathrm{int}(T_j) = \emptyset$, $\cup T_i = Q$ and each vertex of these simplices has only 0/1 coordinates. Prove that $m \leq d!$
Let $m(d) := \min m$. Prove that $m(d) < c^d$ for some absolute constant.
Prove an exponencial lower bound for $m(d)$.

22. Find a sufficient and necessary condition (similar to Farkas' lemma) that $A\mathbf{x} = \mathbf{b}$ has a **non-negative solution**, i.e., with $\mathbf{x} \geq \mathbf{0}$.

23. Find a sufficient and necessary condition (similar to Farkas' lemma) that $A\mathbf{x} = \mathbf{b}$ has a **positive solution**, i.e., with $\mathbf{x} > \mathbf{0}$.