MATH 230 EXAM 1 SPRING 2006

Problem 1: Evaluate the integral

$$I_1 = \int x^3 \, (x^2+1)^{\frac{3}{2}} \, dx.$$

Solution: Use the trigonometric substitution

$$x = \tan\theta \Rightarrow dx = \sec^2\theta \, d\theta$$

and the trigonometric identity

$$\tan^2\theta + 1 = \sec^2\theta$$

to obtain

$$\begin{aligned} I_1 &= \int \tan^3\theta \, \sec^3\theta \, \sec^2\theta \, d\theta \\ &= \int (\sec^2\theta - 1) \, \sec^4\theta \, \tan\theta \, \sec\theta \, d\theta. \end{aligned}$$

Using the substitution

$$u = \sec\theta \Rightarrow du = \sec\theta \, \tan\theta \, d\theta,$$

it follows that

$$\begin{aligned} I_1 &= \int (u^2 - 1) \, u^4 \, du \\ &= \int (u^6 - u^4) \, du \\ &= \left(\frac{u^7}{7} - \frac{u^5}{5} \right) + c \\ &= \left(\frac{\sec^7\theta}{7} - \frac{\sec^5\theta}{5} \right) + c. \end{aligned}$$

From the reference triangle we obtain that

$$\sec\theta = \sqrt{x^2+1}$$

and therefore,

$$I_1 = \frac{(x^2+1)^{7/2}}{7} - \frac{(x^2+1)^{5/2}}{5} + c.$$

Problem 2: Evaluate the integral

$$I_2 = \int \frac{2\, e^t}{1 - e^{2t}}\, dt.$$

Solution: we use the substitution

$$u = e^t \Rightarrow du = e^t\, dt,$$

so the integral becomes

$$I_2 = \int \frac{2}{1 - u^2}\, du.$$

We notice that the denominator of the integrand can be factored:

$$\frac{2}{1 - u^2} = \frac{2}{(1-u)(1+u)},$$

and, since the linear factors in the denominator are distinct, we look for a partial fraction decomposition of the form

$$\frac{2}{(1-u)(1+u)} \quad = \quad \frac{A}{1-u} + \frac{B}{1+u}$$

which implies that

$$\frac{1}{(1-u)(1+u)} = \frac{A(1+u) + B(1-u)}{(1-u)(1+u)} = \frac{(A-B)u + (A+B)}{(1-u)(1+u)}.$$

From this we obtain the system of equations

$$\begin{aligned} A - B &= 0 \\ A + B &= 2 \end{aligned}$$

whose solution is

$$A = B = 1.$$

Therefore,

$$\begin{aligned} I_2 &= \int \left(\frac{1}{1-u} + \frac{1}{1+u} \right) du. \\ &= -\ln|1-u| + \ln|1+u| + c \\ &= \ln\left|\frac{1+u}{1-u}\right| + c \\ &= \ln\left|\frac{1+e^t}{1-e^t}\right| + c \end{aligned}$$

Problem 3: Evaluate the integral

$$I_3 = \int \frac{8\,x^2 + 3\,x + 7}{(x^2+1)(x+1)}\,dx.$$

Solution: we notice that in the denominator of the integrand there is a linear and a quadratic term. Hence, we look for a partial fraction decomposition of the integrand in the form

$$\frac{8\,x^2 + 3\,x + 7}{(x^2+1)(x+1)} = \frac{Ax+B}{x^2+1} + \frac{C}{x+1}.$$

From this it follows that

$$\frac{8\,x^2 + 3\,x + 7}{(x^2+1)(x+1)} = \frac{(Ax+B)(x+1) + C\,(x^2+1)}{(x^2+1)(x+1)} = \frac{(A+B)\,x^2 + (A+B)x + B + C}{(x^2+1)(x+1)}.$$

Therefore, it must hold

$$A+B=8 \quad A+B=3 \quad B+C=7.$$

If we add the first two equations and subtract the third one, we obtain $2\,A = 4$. Hence, we can find the solution to the system:

$$A=2 \quad B=1 \quad C=6,$$

and the integral becomes

$$\begin{aligned} I_3 &= \int \left(\frac{2\,x+1}{x^2+1} + \frac{6}{x+1} \right) dx \\ &= \int \left(\frac{2\,x}{x^2+1} + \frac{1}{x^2+1} + \frac{6}{x+1} \right) dx \\ &= \ln|x^2+1| + \arctan(x) + 6\,\ln|x+1| + c. \end{aligned}$$

Problem 4: Evaluate the integral

$$I_4 = \int \frac{6\,x - 1}{x^2 + 8\,x + 17}\,dx.$$

Solution: it holds that

$$D_x\,(x^2 + 8\,x + 17) = 2\,x + 8,$$

so, we need to find a and b such that

$$6\,x - 1 = a\,(2\,x + 8) + b.$$

It follows that

$$2\,a = 6, \quad b = -1 - 8\,a \Rightarrow a = 3, \qquad b = -25.$$

Therefore,

$$\begin{aligned} I_4 &= \int \frac{3(2\,x + 8) - 25}{x^2 + 8\,x + 17}\,dx \\ &= 3 \int \frac{2\,x + 8}{x^2 + 8\,x + 17} - 25 \int \frac{1}{x^2 + 8\,x + 17}\,dx \\ &= 3\,\ln|x^2 + 8\,x + 17| - 25 \int \frac{1}{(x + 4)^2 + 1}\,dx \\ &= 3\,\ln|x^2 + 8\,x + 17| - 25\,\arctan(x + 4) + c. \end{aligned}$$

Problem 5: Determine whether the integrals below diverge or converge. Justify your answer. In the case of a convergent integral, evaluate it.
(a)

$$I_a = \int_1^\infty \frac{\ln x}{x^2}\,dx,$$

Solution: first we consider the integral

$$I_t = \int_1^t \frac{\ln x}{x^2}\,dx,$$

which can be evaluated using integration by parts, with

$$u = \ln x \Rightarrow du = \frac{dx}{x}$$

and

$$dv = \frac{dx}{x^2} \Rightarrow v = \frac{-1}{x}.$$

Therefore

$$I_t = -\frac{\ln x}{x}\Big|_1^t + \int_1^t \frac{dx}{x^2} = -\frac{\ln t}{t} - \frac{1}{x}\Big|_1^t = -\frac{\ln t}{t} - \frac{1}{t} + 1.$$

Next, we take the limit as $t \to \infty$, it follows that

$$\lim_{t\to\infty} I_t = -\lim_{t\to\infty} \frac{\ln t}{t} - \lim_{t\to\infty} \frac{1}{t} + 1 = \lim_{t\to\infty} \frac{\frac{1}{t}}{1} - 0 + 1 = 1,$$

where we used L'Hopital's rule to evaluate the first limit. Therefore, I_a converges, and has value 1.
(b)

$$I_b = \int_0^2 \frac{1}{x-2}\,dx,$$

Solution: first we consider the integral

$$I_t = \int_0^t \frac{1}{x-2}\,dx = \ln|x-2|\big|_0^t = \ln|t-2| - \ln 2,$$

If we take the limit as $t \to 2^-$, it follows that

$$\lim_{t\to 2^-} I_t = \lim_{t\to 2^-} \ln|t-2| - \ln 2.$$

Since,

$$\lim_{s\to 0} \ln s = -\infty,$$

it follows that the first limit, and hence the integral diverges.