MATH 230 EXAM 2 SPRING 2006

Problem 1: Find the area of the surface of revolution generated by revolving the curve

$$y = \frac{1}{2}(e^x + e^{-x}), \qquad 0 \le x \le 1,$$

around the y-axis.

Solution: the area is given by

$$A = 2\pi \int_0^1 x\sqrt{1 + (y'(x))^2}\, dx.$$

we notice that

$$y'(x) = \frac{1}{2}(e^x - e^{-x})$$

and so,

$$1 + (y'(x))^2 = 1 + \frac{1}{4}(e^{2x} - 2 + e^{-2x}) = \frac{1}{4}(e^{2x} + 2 + e^{-2x}) = \left(\frac{1}{2}(e^x + e^{-x})\right)^2.$$

Hence,

$$\begin{aligned} A &= 2\pi \int_0^1 \frac{1}{2} x\,(e^x + e^{-x})\, dx \\ &= \pi \int_0^1 x\,(e^x + e^{-x})\, dx \\ &= \pi(x\,(e^x - e^{-x})\big|_0^1 - \int_0^1 (e^x - e^{-x})\, dx) \\ &= \pi(e - e^{-1} - (e^x + e^{-x})\big|_0^1) \\ &= \pi(e - e^{-1} - (e + e^{-1}) + (1+1)) \\ &= 2\pi\left(1 - \frac{1}{e}\right) \end{aligned}$$

Problem 2: (a) Solve the initial value problem

$$\frac{dy}{dx} = y + 5, \qquad y(0) = 3.$$

Solution: we solve using separation of variables

$$\frac{dy}{dx} = y+5 \;\rightarrow\; \frac{dy}{y+5} = dx \;\rightarrow\; \ln|y+5| = x+C \;\rightarrow\; y+5 = c\,e^x \;\rightarrow\; y = c\,e^x - 5.$$

Also,

$$y(0) = c - 5 = 3 \;\rightarrow\; c = 8.$$

Therefore,

$$y(x) = 8\,e^x - 5.$$

(b) Determine whether or not the sequence $\{a_n\}$ converges, and find its limit if it does converge.

$$a_n = \frac{\sqrt[4]{n}}{\ln n},$$

Solution: we notice that as $n \to \infty$, $n^{\frac{1}{4}} \to \infty$ and also $\ln n \to \infty$. Therefore, we have to use L'Hopital's rule.

$$\lim_{n\to\infty} \frac{\sqrt[4]{n}}{\ln n} = \lim_{n\to\infty} \frac{\frac{1}{4}\,n^{-\frac{3}{4}}}{\frac{1}{n}} = \frac{1}{4} \lim_{n\to\infty} n^{\frac{1}{4}} = +\infty,$$

hence the sequence diverges.

Problem 3: Determine whether the following series converge or diverge. Justify your answer.
(a)

$$\sum_{n=1}^{\infty} \frac{\sqrt{n}+2}{3\,n+7}$$

Solution: we notice that the series is a positive term series, and we use the limit comparison test, with

$$a_n = \frac{\sqrt{n}+2}{3\,n+7}$$

and

$$b_n = \frac{1}{3\,\sqrt{n}}.$$

It follows that

$$L = \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{\sqrt{n}+2}{3\,n+7}}{\frac{1}{3\,\sqrt{n}}} = \lim_{n\to\infty} \frac{3\,n+6\,\sqrt{n}}{3\,n+7} = \lim_{n\to\infty} \frac{1+\frac{2}{n}}{1+\frac{7}{3\,n}} = 1.$$

But, the series

$$\sum_{n=1}^{\infty} b_n$$

is a p-series, with $p = \frac{1}{2} < 1$, and as such, it diverges. Therefore, by the limit comparison test, since $0 < L < \infty$, it follows that the series

$$\sum_{n=1}^{\infty} \frac{\sqrt{n}+2}{3\,n+7},$$

also diverges.

(b)

$$\sum_{n=1}^{\infty} \frac{3^n}{n^n}$$

Solution: the root test yields

$$\rho = \lim_{n\to\infty} \sqrt[n]{\frac{3^n}{n^n}} = \lim_{n\to\infty} \frac{3}{n} = 0 < 1.$$

Hence, the series converges.

Problem 4: Determine whether the following series converge or diverge. Justify your answer. If they converge, find their sum.
(a)

$$\sum_{n=0}^{\infty} \frac{2^n + 3^n}{7^n}$$

Solution: we notice that

$$\sum_{n=0}^{\infty} \frac{2^n + 3^n}{7^n} = \sum_{n=0}^{\infty} \left(\left(\frac{2}{7}\right)^n + \left(\frac{3}{7}\right)^n \right).$$

Also, it holds that

$$\sum_{n=0}^{\infty} \left(\frac{2}{7}\right)^n = \frac{1}{1-\frac{2}{7}} = \frac{7}{5},$$

since it is a geometric series with $r = \frac{2}{7} < 1$. Similarly,

$$\sum_{n=0}^{\infty} \left(\frac{3}{7}\right)^n = \frac{1}{1-\frac{3}{7}} = \frac{7}{4},$$

since it is a geometric series with $r = \frac{3}{7} < 1$. Therefore, from the theorem on termwise addition of convergent series, it follows that

$$\sum_{n=0}^{\infty} \frac{2^n + 3^n}{7^n}$$

is also convergent, and its sum is

$$\sum_{n=0}^{\infty} \frac{2^n + 3^n}{7^n} = \frac{63}{20}.$$

(b)

$$\sum_{n=3}^{\infty} \frac{1}{n\,\ln(n)}$$

Solution: the series is one with positive terms, and if we define $f(x) = \frac{1}{x\,\ln x}$, then f is a continuous, positive-valued, and decreasing function. So, the integral test is applicable.

$$\int_3^{\infty} \frac{1}{x\,\ln x}\,dx = \lim_{t\to\infty} \int_3^{t} \frac{1}{x\,\ln x}\,dx = \lim_{t\to\infty} \ln(\ln x))|_3^t = \lim_{t\to\infty} \ln(\ln t) - \ln(\ln 3) = +\infty.$$

Therefore, by the integral test, since the integral diverges, the series also diverges.

Problem 5: Determine whether the following series alternating converge or diverge. In case they converge, decide what kind of convergence we have: absolute or conditional. Justify your answer. (a)

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4+1}$$

Solution: we notice that the series

$$\sum_{n=1}^{\infty} \frac{1}{n^4+1}$$

is dominated by the p-series

$$\sum_{n=1}^{\infty} \frac{1}{n^4},$$

which converges, since $p = 4 > 1$. Therefore, since they are both positive term series, by the comparison theorem it follws that

$$\sum_{n=1}^{\infty} \frac{1}{n^4+1}$$

also converges. But, this implies that

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4+1}$$

is absolutely convergent.

(b)

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{1+\frac{1}{n}}$$

Solution: we notice that

$$\lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = 1.$$

Hence the limit

$$\lim_{n\to\infty} \frac{(-1)^n}{1+\frac{1}{n}}$$

does not exist, and therefore, by the n-th term test for divergence, the series diverges.