Problem 1

Suppose that a body moves through a resisting medium so that its deceleration dv/dt is proportional to its velocity v. If it takes the body 10 s to decelerate from an initial velocity $v_0 = 27$ m/s to 10 m/s, find

(a) its velocity after 20 s, and

(b) the total distance it has travelled during the 20 s, assuming that its initial position was $x(0) = 0$.

(a) $\frac{dv}{dt} = -Kv \Rightarrow \frac{1}{v}\frac{dv}{dt} = -k \Rightarrow \ln|v| = -kt + c_1 \Rightarrow$

$v = ce^{-kt}$ $\quad v(0) = c = 27$ $\quad v(10) = 27e^{-10k} = 10 \Rightarrow$

$e^{-10k} = \frac{10}{27} \Rightarrow -10K = \ln\left(\frac{10}{27}\right) = -\ln\left(\frac{27}{10}\right) \Rightarrow k = \frac{1}{10}\ln\left(\frac{27}{10}\right)$

so, $v(t) = 27\, e^{-\frac{t}{10}\ln\left(\frac{27}{10}\right)} \Rightarrow v(20) = 27e^{-\frac{20}{10}\ln\left(\frac{27}{10}\right)} \Rightarrow$

$v(20) = 27\left(\frac{27}{10}\right)^{-2} = 27\left(\frac{10}{27}\right)^2 = 27\,\frac{100}{27^2} = \frac{100}{27}$ m/s

(b) $\frac{dx}{dt} = v \Rightarrow x(t) = \int_0^t v(s)ds + x(0) = \int_0^t ce^{-ks}ds + 0 \Rightarrow$

$x(t) = -\frac{c}{k}e^{-kt} + \frac{c}{k} \Rightarrow x(t) = \frac{27\cdot 10}{\ln\left(\frac{27}{10}\right)}\left(1 - e^{-\frac{t}{10}\ln\left(\frac{27}{10}\right)}\right) \Rightarrow$

$x(20) = \frac{270}{\ln\left(\frac{27}{10}\right)}\left(1 - e^{-2\ln\left(\frac{27}{10}\right)}\right) = \frac{270}{\ln\left(\frac{27}{10}\right)}\left[1 - \left(\frac{10}{27}\right)^2\right]$ m.

Problem 2

Consider a mass-spring-dashpot system having position function $x(t)$ satisfying $4x'' + 4x' + x = 0$.
(a) If $x(0) = 2$ and $x'(0) = -6$, find $x(t)$.
(b) Show that the mass passes through $x = 0$ at some instant $t > 0$.

(a) $4r^2 + 4r + 1 = 0 \Rightarrow (2r+1)^2 = 0 \Rightarrow r = -\frac{1}{2}$ repeated root

$x(t) = (c_1 + c_2 t)e^{-t/2} \Rightarrow x'(t) = c_2 e^{-t/2} + (c_1 + c_2 t)\left(-\frac{1}{2}\right)e^{-t/2}$

$x(0) = c_1 = 2$ and $x'(0) = c_2 - \frac{1}{2}c_1 = -6 \Rightarrow c_2 = -6 + \frac{1}{2}c_1 = -6+1 = -5$

Hence, $x(t) = (2 - 5t)\, e^{-t/2}$.

(b) $x(t_0) = 0 \Rightarrow 2 - 5t_0 = 0$ since $e^{-t/2} > 0$ for all t.

So, when $t_0 = \frac{2}{5}$ $x(t_0) = 0$.

Problem 3

Consider a mass-spring-dashpot system with forcing modelled by $x'' + 4x' + 10x = 74\sin 3t$.
(a) Find the steady periodic oscillations of the system.
(b) Find the transient motion of the system if $x(0) = 0$, $x'(0) = 0$.

(a) $r^2+4r+10=0 \Rightarrow r = \frac{-4 \pm \sqrt{16-40}}{2} = -2 \pm \frac{1}{2}\sqrt{-24} = -2 \pm i\sqrt{6}$

$x_c = e^{-2t}(c_1 \cos\sqrt{6}t + c_2 \sin\sqrt{6}\,t)$

$x_p = A\cos 3t + B\sin 3t \Rightarrow x_p' = -3A\sin 3t + 3B\cos 3t \Rightarrow$

$x_p'' = -9A\cos 3t - 9B\sin 3t$. So, $-9A\cos 3t - 9B\sin 3t - 12A\sin 3t + 12B\cos 3t +$

$10A\cos 3t + 10B\sin 3t = 74\sin 3t \Rightarrow -9A + 12B + 10A = 0$ and

$-9B - 12A + 10B = 74 \Rightarrow A = -12B$ and $B = 74 + 12A \Rightarrow B = \frac{74}{145}$ and

$A = \frac{-12\cdot 74}{145}$. Hence, $x_p = \frac{-888}{145}\cos 3t + \frac{74}{145}\sin 3t$.

(b) $x = x_p + x_c \Rightarrow x = -\frac{888}{145}\cos 3t + \frac{74}{145}\sin 3t + e^{-2t}(c_1\cos\sqrt{6}t +$

$c_2\sin\sqrt{6}t) \Rightarrow x(0) = -\frac{888}{145} + c_1 = 0 \Rightarrow c_1 = \frac{888}{145}$.

$x' = -3\cdot\frac{888}{145}\sin 3t + \frac{3\cdot 74}{145}\cos 3t - 2e^{-2t}(c_1\cos\sqrt{6}t + c_2\sin\sqrt{6}t) +$

$e^{-2t}(-\sqrt{6}c_1\sin\sqrt{6}t + \sqrt{6}c_2\cos\sqrt{6}t) \Rightarrow x'(0) = \frac{3\cdot 74}{145} - 2c_1 + \sqrt{6}c_2 = 0 \Rightarrow$

$\sqrt{6}c_2 = 2\cdot\frac{888}{145} - \frac{3\cdot 74}{145} = (2\cdot 12 - 3)\frac{74}{145} = \frac{21\cdot 74}{145} \Rightarrow c_2 = \frac{21\cdot 74}{\sqrt{6}\cdot 145}$

So, $x_c = e^{-2t}\left[\frac{888}{145}\cos\sqrt{6}t + \frac{1554}{\sqrt{6}\cdot 145}\sin\sqrt{6}t\right]$

Problem 4

(a) Set up an appropriate form of a particular solution y_p WITHOUT determining the coefficients for the equation $y''' + 6y'' + 9y' = x^2 + xe^{-3x}$.

(b) Use the method of variation of parameters to find a particular solution of the equation $y''+3y'+2y = \tan x$. DO NOT evaluate the integrals, but evaluate the Wronskian determinant of the solutions y_1 and y_2 of the corresponding homogeneous problem.

(a) $r^3+6r^2+9r=0 \iff r(r^2+6r+9)=0 \iff r(r+3)^2=0 \Rightarrow r=0, -3$ (repeated)

$y_c = c_1 + c_2 e^{-3x} + c_3 x e^{-3x}$

$y_p = x\left[A_0 + A_1 x + A_2 x^2\right] + x^2\left[B_0 + B_1 x\right] e^{-3x}$

(b) $r^2+3r+2=0 \Rightarrow (r+2)(r+1)=0 \Rightarrow y_c = c_1 e^{-2x} + c_2 e^{-x}$.

$y_1 = e^{-2x}$ $\quad y_2 = e^{-x}$ $\quad W = \begin{vmatrix} e^{-2x} & e^{-x} \\ -2e^{-2x} & -e^{-x} \end{vmatrix} = -e^{-2x}e^{-x} + 2e^{-2x}e^{-x} \Rightarrow$

$W = -e^{-3x} + 2e^{-3x} = e^{-3x}$.

$y_p = -y_1(x)\int \frac{y_2 \tan s}{W}\,ds + y_2(x)\int \frac{y_1 \tan s}{W}\,ds \Rightarrow$

$y_p = -e^{-2x}\int \frac{e^{-s}\tan s}{e^{-3s}}\,ds + e^{-x}\int \frac{e^{-2s}\tan s}{e^{-3s}}\,ds$

$y_p = -e^{-2x}\int e^{2s}\tan s\,ds + e^{-x}\int e^{s}\tan s\,ds$