MATH 230 **EXAM 3** **SPRING 2006**

Problem 1: (a) Find an equation of the ellipse with focus $(1,2)$, directrix $y = 8$ and eccentricity $e = \frac{1}{2}$.

Solution: according to the definition of the ellipse, it holds

$$(x-1)^2 + (y-2)^2 = \frac{1}{4}(y-8)^2 \Leftrightarrow (x-1)^2 + y^2 - 4\,y + 4 = \frac{1}{4}(y^2 - 16\,y + 64) \Leftrightarrow$$

$$(x-1)^2 + \frac{3}{4}y^2 = 12 \Leftrightarrow \frac{(x-1)^2}{12} + \frac{y^2}{16} = 1$$

(b) Find the center and plot the ellipse described in part (a).

Solution: the center of the ellipse has coordinates $(1,0)$, the major axis has length 8 and is parallel to the $y-$ axis, while the minor axis has length $4\sqrt{3}$ and is parallel to the $x-$ axis.

Problem 2: (a) Find an equation of the hyperbola with foci $(1, \pm 4)$, and eccentricity $e = \frac{4}{3}$.

Solution: $c = 4$, and $a = \frac{c}{e} = 3$. Hence, $b^2 = c^2 - a^2 = 7$. The center of the hyperbola is the point $(1, 0)$. It follows that the equation of the hyperbola is

$$\frac{y^2}{9} - \frac{(x-1)^2}{7} = 1.$$

(b) Find the asymptotes and plot the hyperbola described in part (a).

Solution: the asymptotes have equation

$$y = \pm \frac{3}{\sqrt{7}} (x - 1).$$

The hyperbola open upwards and downwards.

Problem 3: (a) Find the RADIUS of convergence of the series

$$\sum_{n=0}^{\infty} \frac{3^n (x-1)^n}{n+2}.$$

Solution: the ratio test yields

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \left| \frac{\frac{3^{n+1}(x-1)^{n+1}}{n+3}}{\frac{3^n (x-1)^n}{n+2}} \right| = 3|x-1| \lim_{n\to\infty} \frac{n+2}{n+3} = 3|x-1|$$

It follows that the radius of convergence is $\frac{1}{3}$.

(b) Find the INTERVAL of convergence of the series in part (a).

Solution: in order to find the interval of convergence we need to check the endpoints of $\frac{2}{3} < x < \frac{4}{3}$.

When $x = \frac{2}{3}$, the series becomes

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n+2}.$$

By the alternating series test, the series converges.

When $x = \frac{4}{3}$, the series becomes

$$\sum_{n=0}^{\infty} \frac{1}{n+2}.$$

By the limit comparison test (using the harmonic series), it follows that the series diverges.

Therefore, the interval of convergence is $[\frac{2}{3}, \frac{4}{3})$.

Problem 4: (a) Use the Maclaurin series of $\sin x$ to obtain a series representation for $\sin(x^3)$.

Solution: we know that

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}.$$

Hence,

$$\sin(x^3) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n+3}}{(2n+1)!}.$$

(b) Use POWER SERIES to evaluate the limit

$$\lim_{x\to 0} \frac{\sin(x^3)}{x^2\,(e^x - 1)}.$$

Solution: we will use the power series

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + ...$$

and the one from part (a)

$$\sin(x^3) = x^3 - \frac{x^9}{6} + \frac{x^{15}}{5!} - ....$$

Then,

$$\begin{aligned}
\lim_{x\to 0} \frac{\sin(x^3)}{x^2\,(e^x - 1)} &= \lim_{x\to 0} \frac{x^3 - \frac{x^9}{6} + \frac{x^{15}}{5!} - ...}{x^2(1 + x + \frac{x^2}{2} + ... - 1)} \\
&= \lim_{x\to 0} \frac{x^3 - \frac{x^9}{6} + \frac{x^{15}}{5!} - ...}{x^3 + \frac{x^4}{2} + ...} \\
&= \lim_{x\to 0} \frac{1 + \frac{x^7}{6} - ...}{1 + \frac{x^2}{2} + ...} \\
&= 1
\end{aligned}$$

Problem 5: (a) Find the Taylor polynomial with remainder for the function

$$f(x) = \sqrt{2\,x+3}$$

with $n = 3$ and $a = 0$.

Solution: one can easily calculate

$$\begin{aligned}
f(x) &= (2\,x+3)^{1/2} \\
f'(x) &= (2\,x+3)^{-1/2} \\
f^{(2)}(x) &= -(2\,x+3)^{-3/2} \\
f^{(3)}(x) &= 3\,(2\,x+3)^{-5/2} \\
f^{(4)}(x) &= -15\,(2\,x+3)^{-7/2}
\end{aligned}$$

So, the Taylor polynomial centered at $a = 0$ with remainder is

$$P_3(x) = 3^{1/2} + 3^{-1/2}\,x - \frac{3^{-3/2}}{2!}\,x^2 + \frac{3 * 3^{-5/2}}{3!}\,x^3 - \frac{15\,(2\,z+3)^{-7/2}}{4!}\,x^4$$

where z is between 0 and x.

(b) Find the Maclaurin series for $f(x) = \frac{1}{x+2}$.

Solution: one can start with the geometric series

$$\sum_{n=0}^{\infty} y^n = \frac{1}{1-y}, \qquad |y| < 1,$$

and set $y = -\frac{x}{2}$. Then, it follows that

$$\sum_{n=0}^{\infty} \frac{(-1)^n\,x^n}{2^n} = \frac{1}{1+\frac{x}{2}} \Leftrightarrow \sum_{n=0}^{\infty} \frac{(-1)^n\,x^n}{2^{n+1}} = \frac{1}{2+x}.$$