

Problem 1

The doomsday vs. extinction model

$$x' = ax\left(\frac{x}{T} - 1\right)$$

may need to be modified so that unbounded growth does not occur then  $x$  is above the threshold  $T > 0$ . The simplest way to do this is to introduce another factor that will have the effect of making  $\frac{dx}{dt}$  negative when  $x > K > T$ . Thus, we consider

$$x' = ax\left(\frac{x}{T} - 1\right)\left(1 - \frac{x}{K}\right).$$

1. Find the equilibria and draw the phase line for this equation.
2. Sketch some representative solutions for initial conditions  $0 < x(0) < T$ ,  $T < x(0) < K$ , and  $x(0) > K$ . (Use the slope field of the equation.)
3. Repeat the previous part for  $K = T$ .

Problem 2

It is sometimes reasonable to assume that the rate at which fish are caught depends on their population  $x$ : the more fish there are, the easier it is to catch them. To include this effect the logistic equation is replaced by

$$x' = ax\left(1 - \frac{x}{K}\right) - Ex.$$

1. Find the two equilibria  $x_1 < x_2$  for this equation if  $E < a$ .
2. A sustainable yield  $x_s = Ex_2$  of the fishery is a rate at which fish can be caught indefinitely. Find  $x_s$  as a function of  $E$ .
3. Determine  $E$  so as to maximize  $x_s$  and thereby find the maximum sustainable yield  $x_m$ .

Problem 3

For each of the following differential equations, find all equilibrium solutions and determine whether they are sinks, sources, or shunts. Also, sketch the phase line.

$$(i) \quad x' = x^3 - 4x; \quad (ii) \quad x' = |1 - x^2|.$$

Problem 4

Each of the following families of differential equations depends on a parameter  $a$ . Sketch the corresponding bifurcation diagrams.

$$(i) \quad x' = x^3 - ax; \quad (ii) \quad x' = x^3 - x - a.$$