

Problem 1

Consider the system $X' = AX$ where A is a 4×4 matrix in canonical form whose eigenvalues are $\pm i$ and $\pm i\sqrt{2}$. Describe its flow on the four-dimensional torus, by identifying it with a square $2\pi \times 2\pi$. Specifically,

- (a) determine whether the overall solution is periodic;
- (b) draw the solution curve that starts at $\theta_1 = 0$ and $\theta_2 = x_1$ and ends when $\theta_1 = 0$ for the fourth time.

Problem 2

Compute the exponentials of the following matrices

$$(a) \quad A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 1 & 4 \end{pmatrix} \quad (b) \quad B = \begin{pmatrix} 1+i & 0 \\ 2 & 1+i \end{pmatrix}.$$

Problem 3

- (a) Find an example of two matrices A, B such that

$$\exp(A) \exp(B) \neq \exp(A + B).$$

- (b) Show that if $AB = BA$, then

$$\exp(A) \exp(B) = \exp(A + B).$$

Problem 4

Let $A(t)$ be a continuous family of 2×2 matrices and let $P(t)$ be the matrix solution to the initial value problem $P' = A(t)P$, $P(0) = P_0$. Show that

$$\det P(t) = (\det P_0) \exp \left(\int_0^t \operatorname{tr} A(s) ds \right).$$

Conclude that if the trace of A is zero, then the determinant of the solution is constant.