CRASH REVIEW OF LIMITS

USEFUL LAWS

- **Constant Law:** If
$$f(x) \equiv C,$$
where C is a constant (so $f(x)$ is a constant function), then
$$\lim_{x \to a} f(x) = \lim_{x \to a} C = C.$$

- **Sum Law:** if both of the limits
$$\lim_{x \to a} f(x) = L,$$
and
$$\lim_{x \to a} g(x) = M$$
exist, then
$$\lim_{x \to a} (f(x) \pm g(x)) = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x) = L \pm M.$$
That is, the limit of a sum is the sum of the limits; the limit of a difference is the difference of the limits.

- **Product Law:** if both of the limits
$$\lim_{x \to a} f(x) = L,$$
and
$$\lim_{x \to a} g(x) = M$$
exist, then
$$\lim_{x \to a} (f(x)\, g(x)) = \lim_{x \to a} f(x) \, \lim_{x \to a} g(x) = L\, M.$$
That is, the limit of a product is the product of the limits.

- **Quotient Law:** if both of the limits
$$\lim_{x \to a} f(x) = L,$$
and
$$\lim_{x \to a} g(x) = M$$
exist and if $M \neq 0$, then
$$\lim_{x \to a} \left(\frac{f(x)}{g(x)} \right) = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \frac{L}{M}.$$
That is, the limit of a quotient is the quotient of the limits, provided that the limit of the denominator is not zero.

- **Root Law**: if n is a positive integer and if $a > 0$ for even values of n, then

$$\lim_{x \to a} \sqrt[n]{x} = \sqrt[n]{a}.$$

- **Substitution Law:** if both of the limits

$$\lim_{x \to a} g(x) = L,$$

and

$$\lim_{x \to L} f(x) = f(L)$$

exist, then

$$\lim_{x \to a} f(g(x)) = f\left(\lim_{x \to a} g(x)\right) = f(L).$$

- **Squeeze Law:** suppose that

$$f(x) \leq g(x) \leq h(x)$$

for all x in some neighborhood of a and also that

$$\lim_{x \to a} f(x) = L = \lim_{x \to a} h(x).$$

Then

$$\lim_{x \to a} g(x) = L$$

as well.

L' HÔPITAL'S RULE

- **Indeterminate Forms** $0/0$**:** the rule is applicable if f and g are continuous at a, if f and g have continuous derivatives, and $g\prime \neq 0$, near a, except perhaps for $x = a$. The rule states that if

$$\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0,$$

and

$$\lim_{x \to a} \frac{f\prime(x)}{g\prime(x)}$$

exists, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f\prime(x)}{g\prime(x)}.$$

- **Indeterminate Forms ∞/∞:** The rule states that if

$$\lim_{x\to a} f(x) = \infty,$$

$$\lim_{x\to a} g(x) = \infty,$$

and

$$\lim_{x\to a} \frac{f\prime(x)}{g\prime(x)}$$

exists, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f\prime(x)}{g\prime(x)}.$$

SOME EXAMPLES

1.

$$\begin{aligned} \lim_{x\to 3} (3\,x^2 + 7\,x - 12) &= \lim_{x\to 3} (3\,x^2) + \lim_{x\to 3} (7\,x) - \lim_{x\to 3} (12) \\ &= 3\ \lim_{x\to 3} (x^2) + 7\ \lim_{x\to 3} (x) - 12 \\ &= 36. \end{aligned}$$

2.

$$\begin{aligned} \lim_{x\to 0} \frac{\sin x}{x} &= \lim_{x\to 0} \frac{\cos x}{1} \\ &= 1. \end{aligned}$$

3.

$$\begin{aligned} \lim_{x\to\infty} \left(1 + \frac{1}{x}\right) &= \lim_{x\to\infty} (1) + \lim_{x\to\infty} \left(\frac{1}{x}\right) \\ &= 1 \end{aligned}$$

4.

$$\begin{aligned} \lim_{x\to\infty} x\ \ln\left(1 + \frac{1}{x}\right) &= \lim_{x\to\infty} \frac{\ln\left(1 + \frac{1}{x}\right)}{\frac{1}{x}} \\ &= \lim_{x\to\infty} \frac{\frac{-1}{x^2}\ \frac{1}{1+\frac{1}{x}}}{\frac{-1}{x^2}} \\ &= \lim_{x\to\infty} \frac{1}{1 + \frac{1}{x}} \\ &= 1 \end{aligned}$$

5.

$$\begin{aligned}
\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x} &= \lim_{x\to\infty}\exp\left(x\,\ln\left(1+\frac{1}{x}\right)\right)\\
&= \exp\left(\lim_{x\to\infty}x\,\ln\left(1+\frac{1}{x}\right)\right)\\
&= e.
\end{aligned}$$