QUIZ 2

Name: UIN:

Evaluate the integral:

$$I_1 = \int \tan^3\theta \sec^3\theta \, d\theta \tag{1}$$

Solution: The integrand contains odd powers of $\tan\theta$ and $\sec\theta$. So, we will save one term $\tan\theta\sec\theta$ in order to form $d(\sec\theta)$, and then use the trigonometric identity

$$\tan^2\theta + 1 = \sec^2\theta$$

to express all remaining powers of $\tan\theta$ in terms of $\sec\theta$. This will prepare the integral for the substitution $u = \sec\theta$.

$$\begin{aligned} I_1 &= \int (\sec^2\theta - 1) \tan\theta \sec^3\theta \, d\theta \\ &= \int (\sec^4\theta - \sec^2\theta) \tan\theta \sec\theta \, d\theta. \end{aligned}$$

We set $u = \sec\theta$, so $du = \tan\theta \sec\theta \, d\theta$. Then,

$$\begin{aligned} I_1 &= \int (u^4 - u^2) \, du \\ &= \frac{1}{5}u^5 - \frac{1}{3}u^3 + c \\ &= \frac{1}{5}\sec^5\theta - \frac{1}{3}\sec^3\theta + c. \end{aligned}$$

Similarly, for

$$I_2 = \int \tan^3\theta \sec^5\theta \, d\theta \tag{2}$$

we have

$$\begin{aligned} I_1 &= \int (\sec^2\theta - 1) \tan\theta \sec^5\theta \, d\theta \\ &= \int (\sec^6\theta - \sec^4\theta) \tan\theta \sec\theta \, d\theta. \end{aligned}$$

After setting $u = \sec\theta$, $du = \tan\theta \sec\theta \, d\theta$, it follows that

$$I_1 = \int (u^6 - u^4) \, du$$

$$
\begin{aligned}
&= \frac{1}{7}u^7 - \frac{1}{5}u^5 + c \\
&= \frac{1}{7}\sec^7\theta - \frac{1}{5}\sec^5\theta + c.
\end{aligned}
$$

Evaluate the integral:

$$I_2 = \int \frac{4}{x^3 + 4\,x}\,dx \tag{3}$$

Solution:

$$
\begin{aligned}
\frac{4}{x^3 + 4\,x} &= \frac{4}{x\,(x^2+4)} \\
&= \frac{A}{x} + \frac{Bx + C}{x^2 + 4} \\
&= \frac{A(x^2+4) + x(Bx + C)}{x^3 + 4\,x} \\
&= \frac{(A+B)x^2 + Cx + 4\,A}{x^3 + 4\,x}.
\end{aligned}
$$

From this it follows that

$$
\begin{aligned}
A + B &= 0 \\
C &= 0 \\
4\,A &= 4
\end{aligned}
$$

or equivalently

$$
\begin{aligned}
A &= 1 \\
B &= -1 \\
C &= 0
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
I_2 &= \int \left(\frac{1}{x} - \frac{x}{x^2+4}\right)\,dx \\
&= \int \frac{1}{x}\,dx - \int \frac{x}{x^2+4}\,dx \\
&= \ln|x| - \frac{1}{2}\ln|x^2+4| + c
\end{aligned}
$$

Similarly, for

$$I_2 = \int \frac{1}{x^3 + x}\,dx \tag{4}$$

we have

$$\begin{aligned}
\frac{1}{x^3+x} &= \frac{1}{x\left(x^2+1\right)} \\
&= \frac{A}{x} + \frac{Bx+C}{x^2+1} \\
&= \frac{A(x^2+1)+x(Bx+C)}{x^3+x} \\
&= \frac{(A+B)x^2+Cx+A}{x^3+x}.
\end{aligned}$$

From this it follows that

$$\begin{aligned}
A+B &= 0 \\
C &= 0 \\
A &= 1
\end{aligned}$$

or equivalently

$$\begin{aligned}
A &= 1 \\
B &= -1 \\
C &= 0
\end{aligned}$$

Therefore,

$$\begin{aligned}
I_2 &= \int \left(\frac{1}{x} - \frac{x}{x^2+1}\right) dx \\
&= \int \frac{1}{x}\, dx - \int \frac{x}{x^2+1}\, dx \\
&= \ln|x| - \frac{1}{2}\ln|x^2+1| + c
\end{aligned}$$