QUIZ 3

Name: UIN:

Evaluate the integrals:

$$I_1 = \int x^3 \, (x^2 - 16)^{\frac{1}{2}} \, dx \tag{1}$$

Solution: we set

$$x = 4 \, \sec \theta,$$

then

$$dx = 4 \, \sec \theta \, \tan \theta \, d\theta$$

and

$$\begin{aligned}
(x^2 - 4^2)^{\frac{1}{2}} &= (4^2 \sec^2 \theta - 4^2)^{\frac{1}{2}} \\
&= (4^2 \, (\sec^2 \theta - 1))^{\frac{1}{2}} \\
&= (4^2 \, \tan^2 \theta)^{\frac{1}{2}} \\
&= 4 \, \tan \theta,
\end{aligned}$$

where we used the identity

$$\sec^2 \theta - 1 = \tan^2 \theta.$$

We can now write

$$\begin{aligned}
I_1 &= \int (4 \, \sec \theta)^3 \, (4 \, \tan \theta) \, (4 \, \sec \theta \, \tan \theta) \, d\theta \\
&= 4^5 \int \sec^4 \theta \, \tan^2 \theta \, d\theta \\
&= 4^5 \int \sec^2 \theta \, (1 + \tan^2 \theta) \, \tan^2 \theta \, d\theta \\
&= 4^5 \int \sec^2 \theta \, (\tan^2 \theta + \tan^4 \theta) \, d\theta
\end{aligned}$$

Next, we set

$$u = \tan \theta$$

which impies that

$$du = \sec^2 \theta \, d\theta.$$

Hence,

$$I_1 = 4^5 \int (u^2 + u^4) \, du$$

$$
\begin{aligned}
&= 4^5 (\frac{u^3}{3} + \frac{u^5}{5}) + c \\
&= 4^5 \left(\frac{\tan^3\theta}{3} + \frac{\tan^5\theta}{5} \right) + c
\end{aligned}
$$

We know that

$$\sec\theta = \frac{x}{4},$$

so from the reference triangle we obtain

$$\tan\theta = \frac{\sqrt{x^2-16}}{4}.$$

Therefore,

$$I_1 = \frac{4^5}{5} \left(\frac{(x^2-16)^{3/2}}{4^3} + \frac{(x^2-16)^{5/2}}{4^5} \right) + c,$$

The integral

$$\int x^3\,(x^2-9)^{\frac{1}{2}}\,dx$$

can be solved similarly.

$$I_2 = \int \frac{2\,x+5}{x^2+4\,x+5}\,dx \tag{2}$$

Solution: We notice that

$$D_x(x^2+4x+5) = 2x+4,$$

so we need to find A and B such that

$$2\,x+5 = A\,(2\,x+4) + B.$$

It follows easily that A and B must satisfy the system of equations

$$
\begin{aligned}
2A &= 2 \\
4A + B &= 5,
\end{aligned}
$$

which is equivalent to

$$
\begin{aligned}
A &= 1 \\
B &= 1.
\end{aligned}
$$

Therefore,

$$I_2 = \int \frac{2x+4}{x^2+4x+5}\,dx + \int \frac{1}{x^2+4x+5}\,dx.$$

For the first integral on the right-hand side we use the substitution

$$u = x^2 + 4x + 5.$$

Then, $du = (2x+4)dx$ and

$$\begin{aligned} \int \frac{2x+4}{x^2+4x+5}\,dx &= \int \frac{1}{u}\,du \\ &= \ln|u| + c_1 \\ &= \ln|x^2+4x+5| + c_1. \end{aligned}$$

For the second integral on the right-hand side, first we complete the square

$$x^2 + 4x + 5 = (x+2)^2 + 1$$

and then we use the substitution

$$v = x + 2$$

Since

$$dv = dx,$$

we obtain

$$\begin{aligned} \int \frac{1}{x^2+4x+5}\,dx &= \int \frac{1}{v^2+1}\,dv \\ &= \arctan(v) + c_2 \\ &= \arctan(x+2) + c_2 \end{aligned}$$

Therefore,

$$I_2 = \ln|x^2+4x+5| + \arctan(x+2) + c.$$

The integral

$$\int \frac{2\,x+7}{x^2+6\,x+10}\,dx$$

can be treated similarly.