QUIZ 4

Name: UIN:

Find the length of the smooth curve $y = \frac{2}{3}(x-1)^{\frac{3}{2}}$ from $x = 1$ to $x = 4$.

Solution: The arc length of a smooth curve is given by

$$s = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

So, the first step is to calculate the derivative

$$\frac{dy}{dx} = (x-1)^{\frac{1}{2}}.$$

Then, since $a = 1$, and $b = 4$ it follows that

$$\begin{aligned} s &= \int_1^4 \sqrt{1 + (x-1)}\, dx \\ &= \int_1^4 \sqrt{x}\, dx \\ &= \frac{2}{3} x^{\frac{3}{2}} \Big|_1^4 \\ &= \frac{2}{3}(4^{\frac{3}{2}} - 1) \\ &= \frac{14}{3} \end{aligned}$$

Similarly, the length of the snooth curve $y = \frac{2}{3}(x-1)^{\frac{3}{2}}$ from $x = 4$ to $x = 9$, is

$$\begin{aligned} s &= \int_4^9 \sqrt{x}\, dx \\ &= \frac{2}{3} x^{\frac{3}{2}} \Big|_4^9 \\ &= \frac{2}{3}(9^{\frac{3}{2}} - 4^{\frac{3}{2}}) \\ &= \frac{38}{3} \end{aligned}$$

Solve the initial value problem $\frac{dy}{dx} = \frac{1}{y^4}$, $\quad y(0) = 1$.
Solution: We use the method of separation of variables to solve the equation.

$$\begin{aligned}
\frac{dy}{dx} &= \frac{1}{y^4} \quad \Rightarrow \\
y^4\, dy &= dx \quad \Rightarrow \\
\frac{1}{5}\, y^5 &= x + c \quad \Rightarrow \\
y^5 &= 5\,x + 5\,c \quad \Rightarrow \\
y &= \sqrt[5]{5\,x + 5\,c}
\end{aligned}$$

The initial condition reads $y(0) = 1$, therefore

$$1 = y(0) = \sqrt[5]{5\,c} \quad \Rightarrow 5\,c = 1 \quad \Rightarrow c = \frac{1}{5}.$$

Finally, the solution is

$$y = \sqrt[5]{5\,x + 1}$$

Similarly, the initial value problem $\frac{dy}{dx} = \frac{1}{y^3}$, $\quad y(0) = 1$, can be solved as

$$\begin{aligned}
\frac{dy}{dx} &= \frac{1}{y^3} \quad \Rightarrow \\
y^3\, dy &= dx \quad \Rightarrow \\
\frac{1}{4}\, y^4 &= x + c \quad \Rightarrow \\
y^4 &= 4\,x + 4\,c \quad \Rightarrow \\
y &= \pm\sqrt[4]{4\,x + 4\,c}
\end{aligned}$$

The initial condition is $y(0) = 1$, therefore

$$1 = y(0) = \pm\sqrt[4]{4\,c} \quad \Rightarrow 4\,c = 1 \quad \Rightarrow c = \frac{1}{4}.$$

Hence, we choose the positive branch, and the solution is

$$y = \sqrt[4]{4\,x + 1}$$