QUIZ 5

Name: UIN:

Determine whether the sequences – whose n-th term is given below – converge or diverge. If they converge, find their limit. Justify your answer.

$$a_n = 1 - \left(-\frac{1}{3}\right)^n, \qquad b_n = \frac{\ln(3\,n)}{\ln(2\,n)}, \qquad c_n = \cos(\pi\,n)$$

Solution: since $\frac{1}{3} < 1$, it follows that

$$\lim_{n\to\infty} \left(-\frac{1}{3}\right)^n = 0.$$

Therefore,

$$\lim_{n\to\infty} a_n = 1 - \lim_{n\to\infty} \left(-\frac{1}{3}\right)^n = 1.$$

For the second one, L' Hopital's rule is needed:

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{\ln(3\,n)}{\ln(2\,n)} = \lim_{n\to\infty} \frac{\frac{3}{3\,n}}{\frac{2}{2\,n}} = 1.$$

Finally, for the third one we notice that when n is even $\cos(\pi\,n) = 1$, while when n is odd, $\cos(\pi\,n) = -1$. Hence the sequence diverges.

Similar results hold for the sequences whose n-th terms are:

$$a_n = 1 - \left(-\frac{1}{4}\right)^n, \qquad b_n = \frac{\ln(4\,n)}{\ln(5\,n)}, \qquad c_n = \sin\left(\frac{\pi\,n}{2}\right)$$

For the first one, we notice that since $\frac{1}{4} < 1$,

$$\lim_{n\to\infty} \left(-\frac{1}{4}\right)^n = 0.$$

Therefore,

$$\lim_{n\to\infty} a_n = 1 - \lim_{n\to\infty} \left(-\frac{1}{4}\right)^n = 1.$$

For the second one, we use L' Hopital's rule:

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{\ln(4\,n)}{\ln(5\,n)} = \lim_{n\to\infty} \frac{\frac{4}{4\,n}}{\frac{5}{5\,n}} = 1.$$

Finally, for the third one we notice that, when n is even $\sin\left(\frac{\pi\,n}{2}\right) = 0$, while when n is odd, $\sin\left(\frac{\pi\,n}{2}\right) = \pm 1$. Hence the sequence diverges.

Determine whether the infinite series below converge or diverge. If they converge, find their sum. Justify your answer.

$$\sum_{n=1}^{\infty} \frac{3\,n}{\sqrt{9\,n^2+5}}, \qquad \sum_{n=1}^{\infty} \frac{3^n}{5^n}$$

Solution: for the first series we notice that

$$\lim_{n\to\infty} \frac{3\,n}{\sqrt{9\,n^2+5}} = \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{5}{9\,n^2}}} = 1 \neq 0.$$

Hence, by the n-th term test for divergence, the series diverges.

The second series is a geometric series with $r = \frac{3}{5} < 1$. Therefore it is convergent, and its sum is:

$$\sum_{n=1}^{\infty} \frac{3^n}{5^n} = \sum_{n=0}^{\infty} \left(\frac{3}{5}\right)^n - 1 = \frac{1}{1-\frac{3}{5}} - 1 = \frac{3}{2}.$$

Similar results hold for the series:

$$\sum_{n=1}^{\infty} \frac{5\,n}{\sqrt{25\,n^2+3}}, \qquad \sum_{n=1}^{\infty} \frac{2^n}{7^n}.$$

For the first one, we notice that

$$\lim_{n\to\infty} \frac{5\,n}{\sqrt{25\,n^2+3}} = \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{3}{25\,n^2}}} = 1 \neq 0.$$

Hence, by the n-th term test for divergence, the series diverges.

The second series is a geometric series with $r = \frac{2}{7} < 1$. Therefore it is convergent, and its sum is:

$$\sum_{n=1}^{\infty} \frac{2^n}{7^n} = \sum_{n=0}^{\infty} \left(\frac{2}{7}\right)^n - 1 = \frac{1}{1-\frac{2}{7}} - 1 = \frac{2}{5}.$$