QUIZ 6

Name: UIN:

Determine whether the infinite series below converge or diverge.

$$\sum_{n=3}^{\infty} \frac{\ln n}{n} \tag{1}$$

Solution: we notice that this is a positive term series, and if we define $f(x) = \frac{\ln x}{x}$, then f is a continuous, positive-valued, and decreasing function. The last property holds because $f'(x) = \frac{1-\ln x}{x^2} < 0$ for $x \geq 3$. So, we can use the integral test, to determine the fate of this series.

$$\begin{aligned}
\int_3^{\infty} \frac{\ln x}{x}\, dx &= \lim_{t\to\infty} \int_3^t \frac{\ln x}{x}\, dx \\
&= \lim_{t\to\infty} \left.\frac{(\ln x)^2}{2}\right|_3^t \\
&= \lim_{t\to\infty} \left(\frac{(\ln t)^2}{2} - \frac{(\ln 3)^2}{2}\right) \\
&= \infty.
\end{aligned}$$

Therefore, by the integral test, since the integral diverges, the series also diverges.

We can deal with the series

$$\sum_{n=1}^{\infty} \frac{n}{e^{n^2}} \tag{2}$$

in a similar way. We notice again that the series is a positive term series, and that if we define $f(x) = \frac{x}{e^{x^2}}$, then f is a continuous, positive-valued, and decreasing function. The last property holds because $f'(x) = \frac{1-2\,x^2}{e^{x^2}} < 0$ for $x \geq 1$. So, the integral test can be used again.

$$\begin{aligned}
\int_1^{\infty} \frac{x}{e^{x^2}}\, dx &= \lim_{t\to\infty} \int_1^t x\, e^{-x^2}\, dx \\
&= \lim_{t\to\infty} \left.\left(-\frac{e^{-x^2}}{2}\right)\right|_1^t \\
&= \lim_{t\to\infty} \left(-\frac{e^{-t^2}}{2} + \frac{e^{-1}}{2}\right)
\end{aligned}$$

$$= \frac{e^{-1}}{2}$$

The last equality holds because

$$\lim_{t\to\infty} e^{-t^2} = 0,$$

Therefore, by the integral test, since the integral converges, the series also converges.

$$\sum_{n=1}^{\infty} \frac{1}{2^n - n} \tag{3}$$

Solution: we notice that this is a positive term series, and if we set $a_n = \frac{1}{2^n - n}$, and $b_n = \frac{1}{2^n}$, then b_n is also positive for all n, and moreover

$$\begin{aligned} L &= \lim_{n\to\infty} \frac{a_n}{b_n} \\ &= \lim_{n\to\infty} \frac{\frac{1}{2^n-n}}{\frac{1}{2^n}} \\ &= \lim_{n\to\infty} \frac{1}{1 - \frac{n}{2^n}} \\ &= 1. \end{aligned}$$

But, the series

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{2^n}$$

is a geometric series with $r = \frac{1}{2} < 1$, so it corverges. Hence, by the limit comparison test both series converge.

The series

$$\sum_{n=1}^{\infty} \frac{1}{3^n - n} \tag{4}$$

can be dealt with in exaclty the same way.