QUIZ 8

Name: UIN:

1. Find the INTERVAL of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{(2\,x)^n}{n}$$

$$\sum_{n=1}^{\infty} \frac{(3\,x)^n}{n}.$$

Solution: We will use the ratio test to find the radius of convergence. We set

$$u_n = \frac{(2\,x)^n}{n},$$

then,

$$\begin{aligned}
\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| &= \lim_{n\to\infty} \left|\frac{\frac{(2\,x)^{n+1}}{n+1}}{\frac{(2\,x)^n}{n}}\right| \\
&= \lim_{n\to\infty} \left|\frac{(2\,x)^{n+1}\,n}{(2\,x)^n\,(n+1)}\right| \\
&= |2\,x| \lim_{n\to\infty} \frac{n}{n+1} \\
&= |2\,x|\,.
\end{aligned}$$

Therefore, according to the ratio test, the power series converges when

$$\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = |2\,x| < 1.$$

This can be written equivalently as $-\frac{1}{2} < x < \frac{1}{2}$, and we need to check what happens at the boundaries of the interval: $x = -\frac{1}{2}$ and $x = \frac{1}{2}$.

When $x = -\frac{1}{2}$, we obtain the alternating series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} = \sum_{n=1}^{\infty} (-1)^n\,a_n.$$

This converges conditionally by the alternating series test, since and $0 < a_{n+1} < a_n$ for all n, and

$$\lim_{n\to\infty} a_n = 0.$$

When $x = \frac{1}{2}$, we obtain the series

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

This is the harmonic series, so it diverges.

Hence, the interval of convergence is $[-\frac{1}{2}, \frac{1}{2})$.

Similarly, the interval of convergence for

$$\sum_{n=1}^{\infty} \frac{(3\,x)^n}{n}$$

is $[-\frac{1}{3}, \frac{1}{3})$.

2. Beginning with the geometric series

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \qquad |x| < 1,$$

find the sum of the series

$$\sum_{n=2}^{\infty} n\,(n-1)\,x^{n+1}, \qquad |x| < 1$$
$$\sum_{n=2}^{\infty} n\,(n-1)\,x^{n+2}, \qquad |x| < 1.$$

Solution: If we differentiate the given geometric series, we obtain the following equation

$$\sum_{n=1}^{\infty} n\,x^{n-1} = \frac{1}{(1-x)^2}.$$

Next, we differentiate both sides of the resulting equation and it follows that

$$\sum_{n=2}^{\infty} n\,(n-1)\,x^{n-2} = \frac{2}{(1-x)^3}.$$

The last step is to multiply both sides by x^3 and x^4, respectively. Therefore,

$$\sum_{n=2}^{\infty} n\,(n-1)\,x^{n+1} = \frac{2\,x^3}{(1-x)^3}, \quad |x| < 1$$
$$\sum_{n=2}^{\infty} n\,(n-1)\,x^{n+2} = \frac{2\,x^4}{(1-x)^3}, \quad |x| < 1.$$