CRASH REVIEW OF SERIES

TWO USEFUL SERIES

1. The geometric series
$$\sum_{n=0}^{\infty} a\,r^n$$
converges if $|r| < 1$ and its sum is
$$\sum_{n=0}^{\infty} a\,r^n = \frac{a}{1-r}.$$
If $|r| \geq 1$ and $a \neq 0$ then the geometric series diverges.

2. The p-series
$$\sum_{n=1}^{\infty} \frac{1}{n^p},$$
converges if $p > 1$ and diverges if $0 < p \leq 1$.

TESTS AND THEOREMS FOR CONSTANT-TERM SERIES

- The n-th term test for divergence tells us that, if either
$$\lim_{n\to\infty} a_n \neq 0$$
or this limit does not exist, then the series
$$\sum a_n$$
diverges. It holds for both positive-term and alternating series.

- The comparison test holds for positive-term series and tells us that $\sum a_n$ converges if $\sum b_n$ converges and $a_n \leq b_n$ for all n; $\sum a_n$ diverges if $\sum b_n$ diverges and $a_n \geq b_n$ for all n. It is useful when one can easily compare the terms of two series, and we know the "fate" of one of them.

- The limit comparison test holds for positive-term series and tells us that if the limit
$$L = \lim_{n\to\infty} \frac{a_n}{b_n}$$
exists and $0 < L < \infty$, then either both series $\sum a_n$, $\sum b_n$ converge, or both diverge. It is useful when one cannot compare very easily the terms of two series, but can easily find this limit. Again, we have to know the "fate" of one of the series.

- The integral test holds for positive-term series and tells us that the series

$$\sum_{n=1}^{\infty} a_n$$

and the integral

$$\int_1^{\infty} f(x)\, dx$$

(where f is a positive-valued, decreasing, continuous function for $x \geq 1$, and such that $f(n) = a_n$ for all integers $n \geq 1$), either both diverge or both converge. It is useful when you can easily integrate f, and/or it's harder to apply the other tests or theorems.

- The alternating series test holds only for alternating series

$$\sum_{n=1}^{\infty} (-1)^{n+1}\, a_n,$$

and tells us that if $a_n \geq a_{n+1} > 0$ for all n and

$$\lim_{n\to\infty} a_n = 0,$$

then the infinite series converges. If either one or both conditions fail, then the test cannot be applied, i.e. we cannot conclude anything about the series.

- The ratio test can be used for both positive-term and alternating series. It says that if the limit

$$\rho = \lim_{n\to\infty} \left| \frac{a_{n+a}}{a_n} \right|$$

exists or is infinite, then the series $\sum a_n$ of nonzero terms, converges absolutely if $\rho < 1$ and diverges if $\rho > 1$. If $\rho = 1$, the ratio test is inconclusive. The ratio test is useful when n-th powers, factorials or combinations of both appear in the expression for a_n, but its use is not limited only to these cases.

- The root test can be used for both positive-term and alternating series. It says that if the limit

$$\rho = \lim_{n\to\infty} \sqrt[n]{|a_n|}$$

exists or is infinite, then $\sum a_n$, converges absolutely if $\rho < 1$ and diverges if $\rho > 1$. If $\rho = 1$, the root test is inconclusive. The root test is useful when a_n looks like $(f(n))^n$, but its use is not limited only to these cases.

BASIC POWER SERIES TO REMEMBER AND KNOW HOW TO DERIVE

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\begin{aligned}
e^x &= \sum_{n=0}^{\infty} \frac{x^n}{n!} \\
\sin x &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2\,n+1}}{(2\,n+1)!} \\
\cos x &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2\,n}}{(2\,n)!} \\
\frac{1}{1-x} &= \sum_{n=0}^{\infty} x^n,\ |x|<1
\end{aligned}
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