MATH 461 - Test 1, Fall 2006

October 6, 2006

Calculators, books, notes and extra papers are *not* allowed on this test

Show all of work to qualify for full credit

1. (20 points) An urn contains 5 red, 7 blue and 2 yellow balls. Three balls are randomly chosen from the urn. Find the probability that they are of the same color if (a) the balls are drawn with replacement, and (b) the balls are drawn without replacement.

   Solution:

   (a)
   $$P(\text{all 3 are of the same color}) = \frac{5^3 + 7^3 + 2^3}{14^3}$$

   (b)
   $$P(\text{all 3 are of the same color}) = \frac{\binom{5}{3} + \binom{7}{3}}{\binom{14}{3}}$$

2. (15 points) A 6 card hand is drawn without replacement from the deck of 52 cards. Find the probability that it contains king and queen of at least one suit.

   Solution: Let

   $$\begin{aligned} A_1 &= \{\text{the hand contains the king and queen of hearts}\}, \\ A_2 &= \{\text{the hand contains the king and queen of spades}\}, \\ A_3 &= \{\text{the hand contains the king and queen of diamonds}\}, \\ A_4 &= \{\text{the hand contains the king and queen of clubs}\}, \text{and} \\ A &= \{\text{the hand contains the king and queen of at least one suit}\}. \end{aligned}$$

   Then $A = A_1 \cup A_2 \cup A_3 \cup A_4$. By the inclusion-exclusion formula,

   $$\begin{aligned} P(A_1 \cup A_2 \cup A_3 \cup A_4) &= \sum_i P(A_i) - \sum_{i<j} P(A_i A_j) + \sum_{i<j<k} P(A_i A_j A_k) - P(A_1 A_2 A_3 A_4) \\ &= \binom{4}{1} P(A_1) - \binom{4}{2} P(A_1 A_2) + \binom{4}{3} P(A_1 A_2 A_3) - P(A_1 A_2 A_3 A_4) \\ &= 4 \cdot \frac{\binom{50}{4}}{\binom{52}{6}} - 6 \cdot \frac{\binom{48}{2}}{\binom{52}{6}} + 4 \cdot \frac{1}{\binom{52}{6}} + 0 \end{aligned}$$

3. (20 points) Urn A contains 2 white balls and 1 black ball, whereas urn B contains 1 white ball and 5 black balls. A ball is drawn at random from urn A and placed in urn B. A ball is then drawn from urn B. (a) Find the probability that the ball drawn from urn B is white. (b) Given that the ball drawn from urn B is white, find the probability that the ball transferred was white.

   Solution:

(a) Let

$$\begin{aligned} H_1 &= \{\text{white ball transferred}\}, \\ H_2 &= \{\text{black ball transferred}\}, \\ A &= \{\text{the ball drawn from urn B is white}\}. \end{aligned}$$

Then

$$\begin{aligned} P(A) &= P(H_1)P(A \mid H_1) + P(H_2)P(A \mid H_2) \\ &= \frac{2}{3} \cdot \frac{2}{7} + \frac{1}{3} \cdot \frac{1}{7} \\ &= \frac{5}{21} \end{aligned}$$

(b)

$$P(H_1 \mid A) = \frac{P(H_1)P(A \mid H_1)}{P(A)} = \frac{\frac{2}{3} \cdot \frac{2}{7}}{\frac{5}{21}} = \frac{4}{5}\,.$$

4. (20 points) The probability of the closing of the ith relay in the circuit below is p_i, where $p_1 = p_2 = 0.5$, $p_3 = 0.8$, and $p_4 = p_5 = 0.7$. Assume that all relays function independently. (a) Find the probability that a current flows from A to B. (b) Given that a current flows from A to B, find the probability that the relay 5 is closed.

Solution: Let $E = \{\text{current flows from A to B}\}$, $A_i = \{i\text{th relay is closed}\}$, $i = 1, 2, 3, 4, 5$.

(a)

$$\begin{aligned} P(E) &= P((A_1 \cup A_2)A_3(A_4 \cup A_5)) \\ &= P(A_1 \cup A_2)P(A_3)P(A_4 \cup A_5) \qquad \text{(by independence)} \\ &= (p_1 + p_2 - p_1 p_2)p_3(p_4 + p_5 - p_4 p_5) \\ &= (0.5 + 0.5 - 0.25)(0.8)(0.7 + 0.7 - 0.49) \\ &= (0.75)(0.8)(0.91) \end{aligned}$$

(b)

$$P(A_5 \mid E) = \frac{P(A_5)P(E \mid A_5)}{P(E)}$$

If the relay 5 is closed, current will flow provided that either relays 1 or 2 are closed, and the relay 3 is closed. Therefore, $P(E \mid A_5) = P((A_1 \cup A_2)A_3)$, implying

$$\begin{aligned} P(A_5 \mid E) &= \frac{P(A_5)P((A_1 \cup A_2)A_3)}{P(E)} \\ &= \frac{P(A_5)P((A_1 \cup A_2)A_3)}{P(A_1 \cup A_2)P(A_3)P(A_4 \cup A_5)} \\ &= \frac{p_5}{p_4 + p_5 - p_4 p_5} = \frac{0.70}{0.91} \end{aligned}$$

5. (10 points) Let X be a random variable whose distribution functions is given by

$$F(x) = \begin{cases} 0\,, & x < -1 \\ x/4 + 1/4\,, & -1 \le x < 0 \\ \frac{1}{2}\,, & 0 \le x < 1 \\ x/12 + 5/6\,, & 1 \le x < 2\,. \\ 1\,, & 2 \le x \end{cases}$$

Find (a) $P(X < 1)$, (b) $P(X = 1)$, (c) $P(0 \le X < 2)$, (d) $P(X > 1/2)$, (e) $P(1 < X \le 5)$.

Solution:

(a) $P(X < 1) = F(1-) = \frac{1}{2}$

(b) $P(X = 1) = p(1) = F(1) - F(1-) = \frac{11}{12} - \frac{1}{2} = \frac{5}{12}$

(c) $P(0 \le X < 2) = F(2-) - F(0-) = 1 - \frac{1}{4} = \frac{3}{4}$

(d) $P(X > \frac{1}{2}) = 1 - \mathbb{P}(X \le \frac{1}{2}) = 1 - F(\frac{1}{2}) = 1 - \frac{1}{2} = \frac{1}{2}$

(e) $P(1 < X \le 5) = F(5) - F(1) = 1 - \frac{11}{12} = \frac{1}{12}$

6. Independent trials, each of which results in a success with probability 1/3, are performed 4 times. Let X be the total number of successes, and let $Y = \cos(\frac{\pi}{2}X)$.

(a) (5 points) Find $P(X \ge 3)$.

(b) (10 points) Find the expectation and the variance of Y.

Solution: X is a binomial random variable with parameters $(4, \frac{1}{3})$.

(a)

$$\begin{aligned} P(X \ge 3) &= P(X = 3) + P(X = 4) = \binom{4}{3}\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right) + \binom{4}{4}\left(\frac{1}{3}\right)^4 \\ &= 4 \cdot \frac{2}{81} + \frac{1}{81} = \frac{1}{9} \end{aligned}$$

(b)

$$\begin{aligned} E[Y] = E[\cos(\frac{\pi}{2}X)] &= \sum_{j=0}^{4} \cos(\frac{\pi}{2})\, P(X = j) \\ &= 1 \cdot P(X = 0) + 0 \cdot P(X = 1) + (-1)P(X = 2) + 0 \cdot P(X = 3) + 1 \cdot P(X = 4) \\ &= \binom{4}{0}\left(\frac{1}{3}\right)^0\left(\frac{2}{3}\right)^4 - \binom{4}{2}\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^2 + \binom{4}{4}\left(\frac{1}{3}\right)^4\left(\frac{2}{3}\right)^0 \\ &= \frac{16}{81} - \frac{24}{81} + \frac{1}{81} = -\frac{7}{81} \end{aligned}$$

$$E[Y^2] = E[\cos^2(\frac{\pi}{2}X)] = \frac{16}{81} + \frac{24}{81} + \frac{1}{81} = \frac{41}{81}$$

$$\mathrm{Var}(Y) = E[Y^2] - (E[Y])^2 = \frac{41}{81} - \left(-\frac{7}{81}\right)^2$$