An SPDE Model of Market Limit Orders

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1 Introduction

This is an introductory paper on modeling of market limit orders based on a type of stochastic moving boundary partial differential equation problems, namely, stochastic Stefan problems, whose existence, in particular, at the boundaries, is justified in [paper that serves as a major part of my Ph.D. thesis].

Recently the emerging trading behaviors and patterns such as high frequency trading (HFT) based on market limit orders tend to have a heavy impact on asset pricing, especially the mechanisms of how the prices are formed, thanks to the rapid technology evolution that brings about faster and faster microprocessors and more and more powerful supercomputers. The classic models [references] of market limit orders are typically static and based on discrete settings. However they seem to have no longer been sufficient to model today's situations which tend to lie in much more dynamic settings than before. We propose a new method of modeling the market limit orders, in which a stochastic heat equation with moving boundaries is used, considering that

(1) The limit orders have a tendency to rapidly smoothen out the jitters which decreases as the price goes beyond the mid-price, according to most of the commonly used HFT algorithms in the finance industry [references].

(2) There is a general preference on the market towards buying or selling of the underlying asset of the orders. For example, when the performance of a particular company did not meet the expectation, traders tend to sell its stocks, thus creating more sell orders than buys; in the opposite, they will be in favor of buying more. This is a general principle but should be better modeled in today's more dynamic settings.

(3) The randomness. Traders (especially those of high frequencies) are constantly creating and canceling limit orders, and the intensity of such behaviors should be in proportion to the existing volumes.

Those basic points will be modeled accordingly using the moving boundary SPDE.

2 Basic Settings

We consider the SPDE model of a market of a predefined set of underlying assets, whose transactions are based on limit orders.

Let t be the time with $t \geq 0$ and S be the quote price from the traders of the specified set of assets. Let $f_A(S,t)$ (resp. $-f_B(S,t)$) be the density of the ask (resp. bid) orders at time t between price S and $S+dS$, i.e., the volume of ask (resp. bid) orders at time t between quote price S and $S+dS$ is $f_A(S,t)dS$ (resp. $-f_B(S,t)dS$) on the limit order book. Here we follow the convention that $f_A \geq 0$ and $f_B \leq 0$. Suppose the mid-price at time t is $S^*(t)$, and all matched orders are executed immediately, which makes perfect sense since all major trading centers today have been computerized, then we have for all (S,t) such that $S > S^*(t)$, $f_B(S,t) = 0$; similarly for all (S,t) such that $S < S^*(t)$, $f_A(S,t) = 0$.

Now for a given (S,t) we decompose the change rate (over time) of the ask density $\partial f_A(S,t)/\partial t$ into several components, which just correspond to our three basic points of the model respectively.

(1) The limit orders have a tendency to rapidly smoothen out the jitters which decreases as the price goes beyond the mid-price, the first component is $\alpha_A(|S - S^*(t)|)\Delta f_A$ where $\alpha_A : [0,\infty) \to (0,\infty)$ is a smooth, monotone decreasing function and $\Delta := \partial^2/\partial S^2$ is the Laplacian. The same to $f_B(S,t)$ with $\alpha_B(\cdot)$ being the same type of functions as $\alpha_A(\cdot)$.

(2) There is a general preference on the market towards buying or selling of the underlying asset of the orders. For example, when the performance of a particular company did not meet the expectation, traders tend to sell its stocks, thus creating more sell orders than buys; in the opposite, they will be in favor of buying more. Such a general preference at a given (S,t) should be in proportion to the existing volumes, so we model this component as $\beta_A(t)f_A(S,t)$ and $\beta_B(t)f_B(S,t)$.

(3) The randomness. Traders are constantly creating and canceling limit orders, and the intensity of such behaviors should be in proportion to the existing volumes at (S,t), so the corresponding component is $\gamma_A f_A(S,t)\dot{W}(S,t)$ and $\gamma_B f_B(S,t)\dot{W}(S,t)$ where $\dot{W} := \partial^2 W/\partial S\partial t$ is a formal derivative and $W(S,t)$ is a 2D Brownian sheet.

In sum, for ask orders, we have for $S \le S^*(t)$, $f_A(S,t) = 0$; for $S > S^*(t)$,

$$\frac{\partial f_A}{\partial t}(S,t) = \alpha_A(|S - S^*(t)|)\frac{\partial^2 f_A}{\partial S^2}(S,t) + \beta_A(t)f_A(S,t) + \gamma_A f_A(S,t)\dot{W}(S,t).$$

Similarly, for buy orders, we have for $S \ge S^*(t)$, $f_B(S,t) = 0$; for $S < S^*(t)$,

$$\frac{\partial f_B}{\partial t}(S,t) = \alpha_B(|S - S^*(t)|)\frac{\partial^2 f_B}{\partial S^2}(S,t) + \beta_B(t)f_B(S,t) + \gamma_B f_B(S,t)\dot{W}(S,t).$$

The change rate of boundary of the two phases, $dS^*(t)/dt$, is determined by how rapidly the sell and buy orders vary around the boundary. Thus, we have

$$\frac{dS^*}{dt}(t) = \rho\left[\frac{\partial f_A}{\partial S}(S^*(t)+,t) + \frac{\partial f_B}{\partial S}(S^*(t)-,t)\right].$$

Note that in practice the price cannot go negative, and sell orders of extremely high price are impossible, we can restrict the problem in a bounded domain $(S,t) \in [0,B] \times [0,T]$.

3 A Transformation and Its Evolution Equation

To remove the effect of a free (moving) boundary between the two phases, we make the following transformation (which will inevitably yield nonlinear terms):

$$\tilde{f}_A(S,t) := f_A(S^*(t) + S, t);$$
$$\tilde{f}_B(S,t) := -f_B(S^*(t) - S, t).$$

Then we have for $0 \le S < B$,

$$\frac{\partial \tilde{f}_A}{\partial t}(S,t) = \alpha_A(S)\frac{\partial^2 \tilde{f}_A}{\partial S^2}(S,t) + \frac{\partial \tilde{f}_A}{\partial S}(S,t)\frac{dS^*}{dt}(t) + \beta_A(t)\tilde{f}_A(S,t) + \gamma_A \tilde{f}_A(S,t)\dot{\tilde{W}}_A(S,t);$$
$$\frac{\partial \tilde{f}_B}{\partial t}(S,t) = \alpha_B(S)\frac{\partial^2 \tilde{f}_B}{\partial S^2}(S,t) - \frac{\partial \tilde{f}_B}{\partial S}(S,t)\frac{dS^*}{dt}(t) + \beta_B(t)\tilde{f}_B(S,t) + \gamma_B \tilde{f}_A(S,t)\dot{\tilde{W}}_B(S,t).$$

with Dirichlet boundary conditions

$$\tilde{f}_A(S,t) = \tilde{f}_B(S,t) = 0, S \in (-\infty, 0] \cup [B, \infty)$$

and Stefan-type condition

$$\frac{dS^*}{dt}(t) = \rho \left[\frac{\partial \tilde{f}_A}{\partial S}(0+,t) - \frac{\partial \tilde{f}_B}{\partial S}(0+,t) \right].$$

Here $\dot{\tilde{W}}_A$ and $\dot{\tilde{W}}_B$ are formal derivatives of the Brownian sheets $\tilde{W}_A(S,t) := W(S^*(t)+S,t)$ and $\tilde{W}_B(S,t) := W(S^*(t)-S,t)$ respectively. Clearly $\tilde{W}_A(S_1,t)$ and $\tilde{W}_B(S_2,t)$ are independent for all positive S_1, S_2, t.

Its evolution equation: for an appropriate kernel $G_t^A(S,s)$ which also encodes $\alpha_A(S)$ and $\beta_A(t)$, we have that for fixed $S \in [0,B], t \geq 0$,

$$\begin{aligned}
\tilde{f}_A(S,t) = & \int_{s=0}^{B} G_t^A(S,s)\tilde{f}_A(s,0)ds \\
& + \int_{r=0}^{t}\int_{s=0}^{B} G_{t-r}^A(S,s)\frac{\partial \tilde{f}_A}{\partial S}(s,r)\frac{dS^*}{dt}(r)dsdr \\
& + \gamma_A \int_{r=0}^{t}\int_{S=0}^{B} G_{t-r}^A(S,s)\tilde{f}_A(s,r)\tilde{W}_A(dsdr).
\end{aligned}$$

Similar to f_B with an appropriate kernel $G_t^B(S,s)$ which also encodes $\alpha_B(S)$ and $-\beta_B(t)$.

Note that these are exactly the type of equations researched in [paper that serves as a major part of my Ph.D. thesis], where we not only show the existence of such an evolution equation and its associated boundary conditions, we further derive an expression of the Stefan boundary, which corresponds to the evolution of the mid-price. Based on these mathematical facts we can further investigate other important quantities such as the evolution of the spread.

4 On-going Research Topics

4.1 Mid-price Formulation

Given the evolution equation, we can then justify the existence of the derivatives of $\tilde{f}_A$ and $\tilde{f}_B$ at $(0+,\cdot)$ using the calculations previously developed, given that we are working on a bounded domain $[0,B]\times[0,T]$. This is a natural extension/application of [paper that serves as a major part of my Ph.D. thesis]. For the formulation of the price, based on that paper, we have

$$\begin{aligned}
\frac{\partial \tilde{f}_A}{\partial S}(0+,t) = & \int_{s=0}^{B} \frac{\partial G_t^A}{\partial S}(0,s)\tilde{f}_A(s,0)ds \\
& + \gamma_A \int_{r=0}^{t}\int_{S=0}^{B} \frac{\partial G_t^A}{\partial S}(0,s)\tilde{f}_A(s,r)\tilde{W}_A(dsdr).
\end{aligned}$$

And we have a similar expression for f_B. Now according to the boundary conditions we have the price evolution as

$$\frac{dS^*}{dt}(t) = \rho \left[\frac{\partial \tilde{f}_A}{\partial S}(0+,t) - \frac{\partial \tilde{f}_B}{\partial S}(0+,t) \right]$$

where $\frac{\partial \tilde{f}_A}{\partial S}(0+,t)$ and $\frac{\partial \tilde{f}_B}{\partial S}(0+,t)$ are defined as above.

4.2 Fitting with Real Data, Estimation, and Prediction

In order to apply the model to the real world we need to discretize with respect to price and volume. In real centralized markets that are order based, the step size of the price, δ, depends on the price of the underlying asset. In our case $\delta = \delta(B)$ is an increasing function of the maximum possible quote price, and should be a step function. Upon discretization we can derive info such as best asks, best bids, ask-bid spreads:

$$A(t) := \delta \inf \left\{ n > S^*(t)/\delta : \int_{S=n\delta}^{(n+1)\delta} f_A(S)dS > K/\delta \right\};$$
$$B(t) := \delta \sup \left\{ n < S^*(t)/\delta : \int_{S=(n-1)\delta}^{n\delta} f_B(S)dS < -K/\delta \right\};$$

and $D(t) := A(t) - B(t)$. Here K is the threshold (step size) for the volume, typically 1.

We can also estimate parameters in our model using different methods such as maximum likelihood, multi-dimensional regression, etc. For example, we have a (portion of) discretized version (sample) of $f_A(S,t)$, $f_B(S,t)$ and $S^*(t)$ from the real data. We may use regression methods to estimate ρ, and analyze the statistics related to the moving boundary, such as variances and covariances. Going from there we can estimate the best law to describe the decay of $\alpha(\cdot)$ (power law, exponential law, etc.) and more importantly, give a robust method to estimate the trend $\beta(\cdot)$ (which should be either by a multi-dimensional regression or a nonparametric method). The estimation of γ can then be done through methods like moments estimation. Other info from the real data, such as total volume, price of maximum volume, ask-bid spread, should also provide guidance to the estimation of model parameters.

For this part, a regression test has been done based on the data extracted from Yahoo! Finance. Note that the results are not good enough to obtain statistically significant ($> 95\%$) conclusions, partly because the data extractor did not work as fast as how data changed (because of the high frequency nature of the limit orders being placed on the markets nowadays). In order to evaluate it more accurately event-driven based commercial transaction data should be needed.

4.3 Simple (balanced) Case

When the market has neither preference towards selling nor buying of the underlying asset (i.e., $\beta_A \equiv 0$; $\beta_B \equiv 0$), and the multiplicative terms of the Laplacians and the Brownian terms are identical respectively, i.e., $\alpha_A \equiv \alpha_B \equiv \alpha$; $\gamma_A = \gamma_B = \gamma$, the evolution equations of the $\tilde{f}_A$ and $\tilde{f}_B$ are same. By exploiting this feature can we simplify the problem to some sort of one-phase problem? The main difficulty should be in the Brownian terms, because most (if not all) of the other terms can be handled by a (further) direct transformation $\tilde{f}(S,t) := \tilde{f}_A(S,t) - \tilde{f}_B(S,t)$. However, one of the main differences is that we do not have nonnegativity here.

A regression test of ρ

Here is a regression test of parameter ρ in the Stefan boundary equation

$$\frac{d}{dt}S*(t)=\rho\left[\frac{\partial f_A}{\partial S}(0+,t)-\frac{\partial f_B}{\partial S}(0+,t)\right]$$

where S is the logarithm of the price and $S*(t)$ is the logarithm of the mid-price. Here I use the logarithm because the price is always positive and our model fits in an unbounded domain (i.e., on a half line).

In reality we approximate $\frac{\partial f_\bullet}{\partial S}(0+,t)$ by $\frac{4F_{1,\bullet}(t)+F_{2,\bullet}(t)}{6\delta}$ (which is a combination of 2/3 of the slope of the volume at the best orders and 1/3 of the slope of the volume at the 2nd best orders) where $F_{1,\bullet}, F_{2,\bullet}$ are the first and second highest (best) asks/bids respectively and δ is the step size of the price with respect to which to place ask and bid orders. Several data manipulation methods are tested and they should provide progressively more accurate regression.

ORIGINAL METHOD: this is the simplest method: pulling data from server (Yahoo finance in our case) for every 10 seconds and take the difference of mid-price divided by 10 directly as an approximation to $\frac{d}{dt}S*(t)$. This method is the easiest one to implement but sometimes (in fact, very often) the mid-price changes too rapidly so that the difference of the mid-price in 10 seconds is not at all a reflection of $\frac{d}{dt}S*(t)$, or it changes too slowly so that we would have a number of useless data pairs with difference of mid-price being zero. In other words, this method is sensitive to how rapid the mid-price moves, which is not ideal for large dataset analysis.

AN IMPROVED METHOD: here for each iteration we repeatedly pull data from the server and records all data until the mid-price changes. Hence we can obtain a more accurate approximation of $\frac{d}{dt}S*(t)$. We use $\frac{4F_{1,\bullet}(t)+F_{2,\bullet}(t)}{6\delta}$ at the beginning, at the end, or a (weighted) average to approximate $\frac{\partial f_\bullet}{\partial S}(0+,t)$. This method needs a balance on how to approximate $\frac{\partial f_\bullet}{\partial S}(0+,t)$ since we do not know at which time point the difference of slopes of order volumes is the real driving force of the corresponding mid-price movement (in real world it could be either at the beginning, at the end, or at a certain time point in the middle) . Another issue is that we have no zero change of mid-price using this algorithm.

A FURTHER IMPROVED METHOD: this is a combination of the preceding methods and is based on the 2nd one (“AN IMPROVED METHOD”). We have a preset timeout value (say 30 seconds) and for each iteration we still pull all the data until the mid-price changes or time is out, whichever comes first. If the mid-price never changes within the timeout period we consider $\frac{d}{dt}S*(t)$ as 0; otherwise we compute the real rate of change of the mid-price over this period of time.

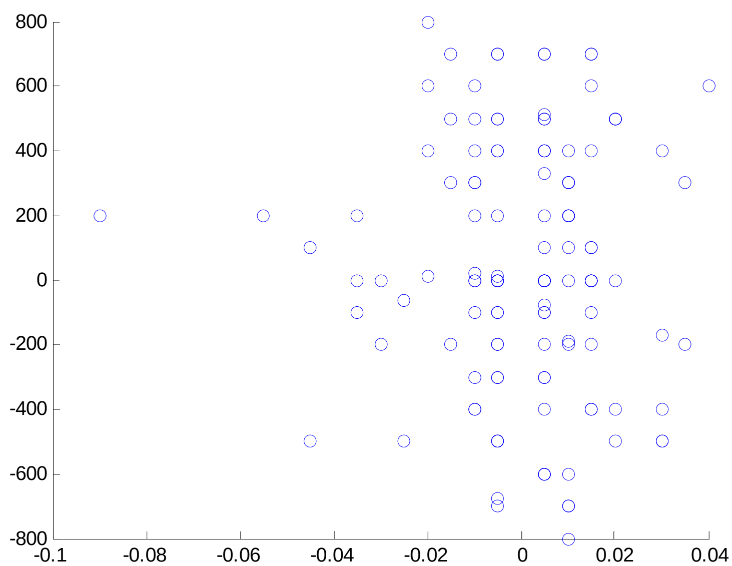
Some Real Results

Currently method 1 and 2 (the original and the improved) are implemented. Method 3 is being implemented and should provide more accurate results. The following results are based on method 2:

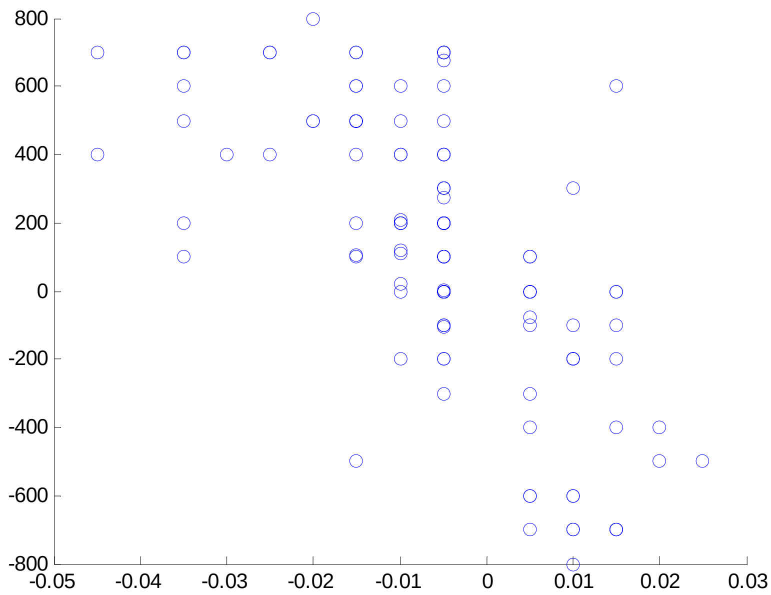
It seems when there is a trend on the mid-price (i.e., it's moving either upwards or downwards) we can have a statistically critical estimation on ρ by a linear regression of the two components. Or at least they are statistically correlated. However when there is no trend (i.e. the mid-price changes randomly without a preference to going up or down) the correlation between the two components gets low and thus a linear regression renders less meaningful.

Also it's worth mentioning that the results obtained by using method 1 (original) are worse and less meaningful than 2 (improved), which should be the consequence of an inaccurate approximation to $\frac{d}{dt}S*(t)$. Method 3 (further improved) is still being implemented and hopefully it can yield some better results. Tweaking how to balance the weight of differences of volume slopes over the period of an iteration should also help, since currently only the last one is used.

(1) This is a scatterplot when the evolution of mid-price has no general trend:



(2) This is a scatterplot when the evolution of mid-price has a clear downward trend:



A linear regression gives

Coefficient = -1.9602e+004 with a 95% confidence interval 1.0e+004 * (-2.3275, -1.5929).

The correlation of the two components is -0.6613.

This means that when there is a clear general trend, the difference between the sizes of ask and bid orders, in particular the 1st and 2nd best orders (combined by a proportion of 2:1), is indeed the driving force of the mid-price movement.